

Relational Representations for Bounded Lattices with Operators

by

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Abstract

Within lattice theory, an interesting question asked is whether a given abstract lattice may be represented concretely as subsets of a closure system on a topological space. This is true for boolean algebras, bounded distributive lattices and arbitrary bounded lattices. In particular, there are a multitude of ways to represent bounded lattices. We present some of these ideas, as well as an analysis of the differences between them. We further investigate the attempts that were made to extend the above representations to lattices endowed with operators, in particular the work done on bounded distributive lattices with operators. We then make a new contribution by extending this work to arbitrary bounded lattices with operators. We also show that the so-called sufficiency operator has a relational representation in the bounded lattice case.

Opsomming

Binne die raamwerk van tralie teorie word die vraag soms gevra of 'n gegewe tralie konkreet verteenwoordig kan word as subversamelings van 'n afsluitings-sisteem op 'n topologiese ruimte. Die voorgenoemde is waar vir, onder andere, boolse algebras, begrensde distributiewe tralies en algemene begrensde tralies. Daar is veral vir begrensde tralies menigte maniere om hul te verteenwoordig. Ons bied sommige van hierdie idees voor, asook 'n analiese van die verskille daarin teenwoordig. Verder ondersoek ons ook sommige van die maniere waarop tralies tesame met operatore verteenwoordig kan word. Ons sal spesiale aandag gee aan distributiewe tralies met operatore, soos gedoen in, met die idee om die voorgenoemde uit te brei na algemene begrensde tralies met operatore. Ons toon dan verder aan dat die sogenoemde voldoende operator ook 'n relasionele verteenwoordiging het in die begrensde tralie geval.

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Chapter 1 - Introduction

In this thesis, we aim to extend the representation for bounded distributive lattices with operators, which was discovered by Goldblatt, [Gol89] to arbitrary bounded lattices. Whereas Goldblatt only considered lattices endowed with join- and meet-hemimorphisms, that is, operators which preserve the join- and meet operations and the top- and bottom elements respectively, we shall also demonstrate a representation for the so-called sufficiency operator (the operator which sends joins to meets and the top to the bottom)- all the while remaining within the framework of the topological duality for bounded lattices developed by Urquhart [Urq78]. We should make clear the distinction between a representation and a duality. By a duality we mean a full dual equivalence of categories. By a representation we mean ‘half’ of a duality, in the sense that the objects can be represented concretely, but not necessarily the morphisms.

Duality theory is a very interesting and important branch of mathematics. It allows one to travel effortlessly from one field of mathematics to another. Many mathematicians spend hours labouring over possible propositions and conjectures in one field, never realising that there are powerful results available in a different field, and that using the methods of duality they would be able to solve their problems quickly. One of the pioneers in duality theory was Marshall Stone, who developed a full duality between boolean algebras and a class of topological spaces [Sto36], which are sometimes referred to as Stone spaces. In the process, he generalized the ideas put forth by Tarski [Tar29], who showed that a complete atomic boolean algebra is isomorphic to the boolean algebra resulting from a powerset. We will omit the work done by Tarski and focus only on Stone’s work - which we study in detail in Chapter 2.

In fact, Stone discovered duality theorems [Sto37] for structures more general than boolean algebras (such as bounded distributive lattices), but it was noticed by Priestley that the duality would be greatly simplified if a partial order is introduced onto the topological space. She thus obtained a very elegant duality for bounded distributive lattices [Pri70]. It is then natural to ask if this can be taken even further to the case of arbitrary bounded lattices. Indeed this question was asked and answered by Urquhart [Urq78]. Since the axiom of choice is required for Priestley’s (and therefore also Urquhart’s) work, some mathematicians wondered if it would be possible to obtain the duality without it. The literature thus splits into three camps: those who follow the approach of Urquhart, those who follow Hartonas [Har97] and the modern approach taken by Moshier and Jipsen [Mos09]. We examine all of these attempts in subsequent sections. Urquhart’s work is a direct generalization of Priestley’s results, and therefore specializes to the distributive case very naturally. Unfortunately the same cannot be said of Hartonas’ results. In their paper Moshier and Jipsen show that we may present and solve the problem of duality in a

more pure form, that is, without considering a subclass of the class of lattices, and without generalizing the class of topological spaces.

After Priestley discovered her duality, work was done by Goldblatt [Gol89] in order to try and understand how the duality would behave if certain structure-preserving maps were added to the lattice. His work followed that of Jónson and Tarski [JoT51] [JoT52], who worked on boolean algebras with operators. In particular, the notion of associating a relation to an operator is due to these authors. Goldblatt then showed that endowing the topological space with binary relations which satisfy various properties is the correct idea to represent unary operators on distributive lattices. We shall look for analogous results in the case where the lattice is still bounded, but not necessarily distributive. We shall see that applying Goldblatt's ideas together with viewing a bounded lattice as the 'combination' of two meet-semilattices provide the key insight to this problem.

In order to understand the methods and important ideas, it is necessary that the reader should be *au fait* with some topological- and order-theoretic concepts. For those readers who are not, the author has attempted to provide as much of the necessary material as possible. Indeed Section 1.2 covers the vast majority of the needed order-theoretic results, and Appendix A covers the needed topological knowledge. For further reading on order theory we refer the interested reader to [Dav90]. A standard topology text is [Kel75].

As a final note, the author is aware that many of the representation results which will be discussed indeed extend to a full duality. We shall mention when this extension is possible, but we shall not delve into the details. We beg the reader's patience in this regard.

1.1 Notation and Terminology

We introduce here a few of the notational conventions which we will find useful or elucidatory. In this thesis the main idea is of course to relate topological concepts and structures to algebraic concepts and structures. However, these structures often have many different substructures, operators and relations defined on them. Keeping track of every little detail can cloud the issue considerably and make the work appear more convoluted and complex. For instance, if X is a set and Ω is a topology for X then technically we should refer to the topological space as the pair (X, Ω) . However, when dealing with topological spaces, we will very frequently refer to X as a topological space when the context is clear, omitting the mention of Ω . This convention is very helpful especially when one encounters the spaces in chapter 4 and onwards, where we study spaces endowed with two pre-orders and various relations.

The same convention holds also for the lattices that we study. Of course a general algebraic lattice is actually a 3-tuple (L, \wedge, \vee) with L a set and \wedge and \vee two binary operations on L . However, we will again refer simply to the lattice as L when the context is clear. In later sections we will meet lattices with a top- and bottom element, complementation and various structure-preserving maps. In all of these cases the convention remains: We refer simply to the lattice as L .

We shall also, by the very nature of duality, be constructing, in a canonical way, structures which are based on different structures. For instance, in the chapter on Stone Duality, we construct a boolean algebra given a topological space that is compact and totally disconnected. Our terminology will work as follows: Suppose that A and B are different mathematical structures and that f is a canonical way of constructing B , given A . We can think pictorially of the situation in the following manner: $f : A \mapsto B$. As we mentioned above, we will be concerned with topological and algebraic structures in this thesis. It will thus happen frequently that we 'convert' a topological space X into a particular lattice via the above philosophy. We will then denote that lattice by L_X to emphasize this fact and we will refer to it as the *dual lattice* of X . Similarly, if we had converted a lattice L into a topological space then we will denote that space by X_L and refer to it as the *dual space* of L .

Further, when we write $A \subseteq B$ then we mean A is contained in B , and is possibly equal to B . $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

1.2 Preliminaries

In addition to the topological results in Appendix A, we require some knowledge about order theory. We state some of the basic definitions and prove some of the results that we will use in the later sections. These results may be found in any decent book on lattice theory (see [Dav90] for example).

Definition 1.1 A partially ordered set (L, \leq) is called a lattice if any two elements $a, b \in L$ have a supremum and an infimum (denoted $a \vee b$ and $a \wedge b$ respectively) in L .

From now on, we refer to $a \vee b$ as the join of a and b and $a \wedge b$ as the meet of a and b .

Definition 1.2 Let L and K be lattices. Then a mapping $f : L \longrightarrow K$ is called a lattice homomorphism (or a lattice morphism) if for all $a, b \in L$

$$\begin{aligned} f(a \vee b) &= f(a) \vee f(b) \quad \text{and} \\ f(a \wedge b) &= f(a) \wedge f(b). \end{aligned}$$

A lattice isomorphism is a bijective lattice homomorphism, and a lattice embedding is an injective lattice homomorphism.

Note that in the above definition we have abused notation and used the symbols \vee and \wedge for both L and K .

Definition 1.3 A lattice L is said to be distributive if it satisfies the distributive law, i.e. if $a, b, c \in L$ implies that

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c), \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c). \end{aligned}$$

Definition 1.4 A lattice L is said to be bounded if there exist elements 0 and 1 such that for any $a \in L$, both $0 \leq a$ and $a \leq 1$ hold. 0 is called the bottom element and 1 the top element of L .

Definition 1.5 Let L and K be bounded lattices. Then a mapping $f : L \longrightarrow K$ is called a bounded lattice homomorphism (or a bounded lattice morphism) if

1. f is a lattice homomorphism and
2. $f(0) = 0$ and $f(1) = 1$.

A boolean algebra isomorphism is a lattice isomorphism that preserves the top and bottom.

Again we have abused notation and used 1 and 0 for the top and bottom of both L and K .

Definition 1.6 A lattice B is called a boolean algebra if it is a bounded distributive lattice with the additional property that for any $b \in B$, there exists an element $b' \in B$, called the complement of b , such that $b \vee b' = 1$ and $b \wedge b' = 0$.

It can be shown that, in a bounded distributive lattice, the complement of an element is unique, so that it makes sense to denote the complement of b by b' . It is perhaps appropriate to consider a few examples of boolean algebras at this juncture. Note that in the example below, and throughout the thesis, $X \setminus A$ will mean the set-theoretic complement of A relative to X .

Example 1.7 All of the boolean algebras we will consider will involve sets.

1. Given any set X , its powerset defines a boolean algebra if we associate the complement in the boolean algebra with the set-theoretic complement, i.e. $\forall A \subseteq X, A' = X \setminus A$. If we further see that $\emptyset = 0$ and $X = 1$, it becomes trivial to show that we then have the boolean algebra $(\mathcal{P}(X), \cap, \cup, ', \emptyset, X)$.
2. The family of all clopen subsets of a topological space $(X; \Omega)$ is also a boolean algebra. Indeed, this family is bounded since X and \emptyset are clopen. Distributivity follows from the fact that they are sets. Lastly, the complement of a clopen set is clopen and obviously unique.
3. The finite-cofinite algebra is defined to be $FC := \{A \subseteq X \mid A \text{ is finite or } X \setminus A \text{ is finite}\}$. It can be shown that $FC(\mathbb{N})$ is an infinite boolean algebra which is not isomorphic to $\mathcal{P}(X)$ for any X .

Definition 1.8 A subset A of a lattice L is called an up-set if for all $a \in A$ and $b \in L$, $a \leq b$ implies that $b \in A$. Dually, A is called a down-set if for all $a \in A$ and $b \in L$, $a \geq b$ implies that $b \in A$.

Later on we shall consider sets with multiple orderings on them. It is then insufficient to use the term ‘up-set’ since there is ambiguity about which ordering this is in reference to. For example, for a set with two pre-orders (L, \leq_1, \leq_2) we shall call $A \subseteq L$ \leq_1 -increasing if it is an up-set with respect to the \leq_1 ordering, and so on. Central to the construction of Stone duality is the idea of a filter.

Definition 1.9 Let L be a lattice. A non-empty subset F of L is called a filter in L if

1. $a, b \in F \Rightarrow a \wedge b \in F$
2. F is an up-set.

An ideal is defined dually, i.e. a down-set which is closed under finite joins.

Example 1.10 Let L be a lattice and let $a \in L$. Define $\uparrow a := \{b \in L \mid a \leq b\}$. Then $\uparrow a$ is a filter in L . Dually, $\downarrow a := \{b \in L \mid a \geq b\}$ is an ideal in L .

Definition 1.11 Let L be a lattice and F a proper filter of L , that is, $F \subset L$. Then F is said to be a prime filter if $a, b \in L$ and $a \vee b \in F \Rightarrow a \in F$ or $b \in F$. The (possibly empty) collection of prime filters of L is denoted $\mathcal{F}_p(L)$. A prime ideal is defined dually, and the collection of prime ideals of L is denoted $\mathcal{I}_p(L)$.

Recall that $\uparrow a = \{b \in L \mid a \leq b\}$, for some element a in a lattice L . This is an example of a filter of a lattice (in fact, it is called the principal filter generated by a).

Lemma 1.12 Let F be any filter in a bounded distributive lattice L and let $a \in L$. Then the set $F_a := \uparrow \{a \wedge c \mid c \in F\}$ is a filter containing F and a .

Proof. Since F_a is an up-set, it is up-closed by definition. Let $x, y \in F_a$. Then $x = a \wedge c_1$ for some $c_1 \in F$. Similarly $y = a \wedge c_2$ for some $c_2 \in F$. Hence $x \wedge y = (a \wedge c_1) \wedge (a \wedge c_2) = a \wedge (c_1 \wedge c_2)$. But F is a filter, thus $c_1 \wedge c_2 \in F$, hence $a \wedge (c_1 \wedge c_2) = x \wedge y \in F_a$. Thus F_a is a filter. Further, since F is up-closed, $1 \in F$. Thus $a = a \wedge 1 \in F_a$. Finally, if $c \in F$, $a \wedge c \in F_a$, but $a \wedge c \leq c$ so that $c \in F_a$. Hence $F \subseteq F_a$. \square

Lemma 1.13 In a distributive lattice with 0, every maximal filter is prime.

Proof. Let F be a maximal filter in a distributive lattice L with 0 and let $a, b \in L$. Assume $a \vee b \in F$ and $a \notin F$. We require $b \in F$. Define $F_a := \uparrow \{a \wedge c \mid c \in F\}$. Then, by Lemma 1.12, F_a is a filter containing F and a . Since F is maximal, $F_a = L$. In particular, $0 \in F_a$, so $0 = a \wedge d$ for some $d \in F$. Then $(a \vee b) \wedge d = (a \wedge d) \vee (b \wedge d) = 0 \vee (b \wedge d) = b \wedge d$. Thus $b \wedge d \in F$ since $(a \vee b) \wedge d \in F$. But $b \wedge d \leq b$ so that $b \in F$. \square

The following is a corollary of Zorn's lemma, which we state without proof.

Lemma 1.14 Let \mathcal{S} be a non-empty family of sets such that $\bigcup_{i \in I} A_i \in \mathcal{S}$ whenever $\{A_i\}_{i \in I}$ is a non-empty chain in $(\mathcal{S}; \subseteq)$. Then \mathcal{S} has a maximal element.

Lemma 1.15 [Dav90] Given a bounded distributive lattice L and an ideal J and a filter G in L such that $J \cap G = \emptyset$, there exists $F \in \mathcal{F}_p(L)$ such that $G \subseteq F$ and $F \cap J = \emptyset$

Proof. Let L be a bounded distributive lattice and G and J a filter and ideal respectively such that they have empty intersection. Define $\mathcal{S} = \{K \mid K \supseteq G \text{ and } K \cap J = \emptyset \text{ where } K \text{ is a filter.}\}$. The set contains G , and so is non-empty. Let $\{C_i\}_{i \in I}$ be a chain in \mathcal{S} . To apply Lemma 1.14, we require that $C := \bigcup_{i \in I} C_i \in \mathcal{S}$. Thus we must show that C is a filter containing G and that it is disjoint from J . Since each C_i contains G , it follows that C also contains G . Further, because each C_i is up closed - all of them being filters - we have that C is up-closed. Also, if $x \in C \cap J$, then $x \in C_i \cap J$ for some $i \in I$, which is impossible. Hence C is disjoint from J . It remains to show that $x, y \in C \Rightarrow x \wedge y \in C$. Now $x \in C \Rightarrow \exists i_1$ such that $x \in C_{i_1}$. Similarly, $y \in C_{i_2}$ for some $i_2 \in I$. But $\{C_i\}_{i \in I}$ is a chain. Assume therefore without loss of generality that $C_{i_1} \subseteq C_{i_2}$. Thus $x, y \in C_{i_2}$. Hence $x \wedge y \in C_{i_2} \subseteq C$. This means that $C \in \mathcal{S}$. Thus, \mathcal{S} has a maximal element F , which by hypothesis also contains G . By Lemma 1.13 F is prime. \square

Corollary 1.16 Given a proper filter G in a bounded distributive lattice L , there exists a prime filter F of L , such that $G \subseteq F$.

Proof. Since G is a proper filter there is an element $a \notin G$. Then $G \cap \downarrow a = \emptyset$. Then we apply Lemma 1.15 to find the prime filter we require. \square

Corollary 1.17 A subset A of a bounded distributive lattice L is contained in a prime filter iff it has the finite meet property, i.e. no finite subfamily of A satisfies $a_1 \wedge \cdots \wedge a_n = 0$.

Proof. Let A be a subset of a bounded distributive lattice L . Let A be contained in a prime filter F , but suppose that A does not have the finite meet property. Thus, $a_1 \wedge \cdots \wedge a_n = 0$ for some finite number of elements $a_i \in A$. But, then $0 \in F$, since $A \subseteq F$ and F is closed under finite intersection. This contradicts the fact that F is proper.

Conversely, suppose that A has the finite meet property. Consider the set $J = \{b \in L \mid a_1 \wedge \cdots \wedge a_n \leq b \text{ for some } a_1, \dots, a_n \in A\}$. Then J is a filter. Indeed, it is obviously up-closed, and if $x, y \in J$, then $a_1 \wedge \cdots \wedge a_n \leq x$ and $b_1 \wedge \cdots \wedge b_m \leq y$. Thus $a_1 \wedge \cdots \wedge a_n \wedge b_1 \wedge \cdots \wedge b_m \leq x \wedge y$. Also, $A \subseteq J$ since \leq is reflexive. Now J has to be proper, since otherwise we would have $a_1 \wedge \cdots \wedge a_n = 0$, which contradicts the fact that A had the finite meet property. Thus we may apply Corollary 1.16 to J , and obtain the prime filter containing A . \square

We can also define the meet and join of arbitrary subsets of a lattice L , not just between two elements. Indeed, for $A \subseteq L$, define $\bigvee A := \sup\{A\}$ and $\bigwedge A := \inf\{A\}$. Note that these suprema and infima may not be in L . This motivates the next definition.

Definition 1.18 A lattice L is called complete if it is closed under arbitrary meets and joins, that is, if for any $A \subseteq L$, both $\bigvee A$ and $\bigwedge A$ are in L .

Definition 1.19 A lattice C is a completion of a lattice L if C is complete and L may be embedded in C .

The natural question to ask is whether or not any (bounded) lattice may be embedded in a complete lattice. The answer to this question is yes, and we shall shortly state the theorem that guarantees this (in fact our theorem will say something stronger). Before that though, we need a few more definitions. We adopt some of the notation of Moshier and Jipsen [Mos09] for the next definition.

Definition 1.20 Let K be a complete lattice and C a sublattice of K . Then C is said to be compact-dense in K if the following two conditions hold.

1. (compactness) For any $a \in C$ and for any family $a_i \in L$ where $i \in I$, if $\bigwedge_{i \in I} a_i \leq a$, then there is a finite subset $F \subset I$ such that $\bigwedge_{i \in F} a_i \leq a$.
2. (density) If $a \in K$, then there exist elements $c_i \in C$, where $i \in I$ and $a \leq c_i$ for each i , such that $a = \bigwedge_{i \in I} c_i$.

Definition 1.21 A completion C of a lattice L is a canonical extension of L if L is compact-dense in C .

We may finally prove the theorem regarding the existence of canonical extensions, first proved by Gehrke and Harding in [Geh01].

Theorem 1.22 Every lattice L has a canonical extension, denoted by L^σ , unique up to isomorphism. That is, if C is also a canonical extension of L , then there is an isomorphism from L^σ to C that keeps L fixed.

We now introduce some order-theoretic properties that are related to topological spaces.

Definition 1.23 Let X be a topological space. Then there is a pre-order on X , called the specialization order, with $x \leq y$ if and only if the closure of $\{x\}$ is contained in the closure of $\{y\}$.

We note at once that, because the one point sets are closed in a T_1 space, the specialization order is only meaningful (non-discrete) on spaces with separation T_0 or lower. An alternative characterization of specialization is that $x \leq y$ iff every open neighbourhood of x is also an open neighbourhood of y .

Definition 1.24 A set D is called a directed subset of a poset X if it has non-empty intersection with $\uparrow x \cap \uparrow y$ for any elements $x, y \in D$.

Definition 1.25 A poset X is called a directed complete partially ordered set (dcpo) if every directed subset of X has a supremum.

Chapter 2 - Stone Duality

2.1 Introduction

In this chapter we will arrive at a topological duality for boolean algebras. As we stated in the introduction, this work is well known in the literature. Indeed, the seminal papers of Marshall Stone [Sto36, Sto37] were written in 1936 and 1937 respectively. They have served as a guiding light for many of the representation results that followed. Recalling the philosophy outlined in the introduction, we need a way to construct a topological space from a given boolean algebra, and conversely to construct a boolean algebra given a suitable topological space. Let us try going from boolean algebras to topological spaces first. Let B be a boolean algebra. There are a few questions we need to answer. For instance, we need to know what the underlying set of this space needs to be. It turns out that the underlying set for this space, called the dual space of B , is to be the set of all prime filters of B . At first glance it seems that this is quite an arbitrary choice, but Stone's original idea was to model an element b of a boolean algebra by the set of boolean algebra homomorphisms h into the two-element boolean algebra such that $h(b) = 1$ where 1 is of course the top element. These homomorphisms are then completely determined by the preimage $h^{-1}(1)$, which can be shown to be a prime filter of B . This is the primary motivation for selecting the prime filters as the underlying set for the dual space. The results and definitions that follow have been taken from [Dav90].

2.2 Stone Spaces

The natural question arises: How do we know there are sufficiently many prime filters for this to be a reasonable choice? How do we know there are any prime filters at all? Lemma 1.15 and Corollary 1.16 guarantee that a boolean algebra is richly supplied with prime filters.

Now, how should we choose the open sets in $\mathcal{F}_p(B)$, the set of all prime filters of B ?

Consider the following. Let $X_a := \{F \in \mathcal{F}_p(B) \mid a \in F\}$. We stated in Example 1.5 that the clopen subsets of a topological space forms a boolean algebra. As we will show, by choosing as a basis for our topology the set $\mathcal{B} := \{X_a \mid a \in B\}$, we will obtain the desired correspondence between clopen sets and members from B .

We will call (X_B, Ω_B) the dual space of B , where $X_B := \mathcal{F}_p(B)$ and Ω_B is the topology formed by choosing \mathcal{B} as a basis.

NOTE: The reader should not confuse X_a with X_B .

Firstly, note that each X_a is clopen. Indeed, it is open by definition and closed since $X \setminus X_a = X_{a'}$, which is open. What we would like is if *all* clopen sets were of the form X_a for some $a \in B$. We shall discover that this is the case in a Stone space. We thus define Stone spaces, and prove that the dual space of a boolean algebra is indeed a Stone space.

Definition 2.1 A topological space (X, Ω) is called a Stone space if it is compact and totally disconnected, i.e. $x, y \in X$ with $x \neq y$ implies there exists a clopen set U such that $x \in U$ and $y \notin U$.

Finally, we may prove that the dual space of a boolean algebra is a Stone space.

Lemma 2.2 Let B be a boolean algebra and (X_B, Ω_B) the dual space of B . Then X is compact.

Proof. Let \mathcal{U} be an open covering of X , and suppose that \mathcal{U} has no finite subcover. Since each element of \mathcal{U} consists of basic open sets, we may regard \mathcal{U} as consisting of basic open sets, i.e. $\mathcal{U} = \{X_a \mid a \in A\}$ for some $A \subseteq B$. By hypothesis, for all finite subsets $\{a_1, \dots, a_n\}$ of A , $X_{a_1} \cup \dots \cup X_{a_n} \neq X$. But $X_{a_1} \cup \dots \cup X_{a_n} = X_{a_1 \vee \dots \vee a_n}$. Indeed, since filters are up-sets and $a \leq a \vee b$, $X_a \cup X_b \subseteq X_{a \vee b}$. However, in a prime filter F , $a \vee b \in F$ implies $a \in F$ or $b \in F$. Thus $X_{a \vee b} \subseteq X_a \cup X_b$. Hence $X_{a_1 \vee \dots \vee a_n} \neq X_1$ (noting that $X = X_1$). Thus $a_1 \vee \dots \vee a_n \neq 1$. By De Morgan's laws, $a'_1 \wedge \dots \wedge a'_n \neq 0$. Thus $A^* := \{a' \mid a \in A\}$ has the finite meet property, and is therefore contained in a prime filter F by Corollary 1.17. But then $F \in X$, and is disjoint from every element in \mathcal{U} , which is a contradiction. Thus, X is compact. \square

Lemma 2.3 Let (X, Ω_B) be the dual space of a boolean algebra B . Then the clopen sets are exactly the sets X_a , for $a \in B$. Further, X is totally disconnected.

Proof. Let U be an arbitrary clopen set of X . Since U is open, $U = \bigcup_{a \in A} X_a$, for some $A \subseteq B$. But U is also closed, and since every closed subset of a compact space is compact, there is a finite subset A_1 of A such that $U = \bigcup_{a \in A_1} X_a$. Thus $U = X_{a_0}$ where $a_0 = \bigvee A_1$. Finally, suppose $F_1, F_2 \in X$ with $F_1 \neq F_2$. Suppose without loss of generality that $a \in F_1$, but $a \notin F_2$. Then $F_1 \in X_a$ but $F_2 \notin X_a$. \square

The following lemma is a useful result, which will assist us in the proof of Theorem 2.7.

Lemma 2.4 Let (X, Ω) be a Stone space. Then

1. Let Y be a closed subset of X and $x \notin Y$. Then there exists a clopen set V such that $Y \subseteq V$ and $x \notin V$.
2. Let Y and Z be disjoint closed subsets of X . Then there exists a clopen set U such that $Y \subseteq U$ and $Z \cap U = \emptyset$.

Proof. For (1) Let $y \in Y$. Then, since X is a Stone space, there exists a clopen set V_y containing y but $x \notin V_y$. The family $\{V_y \mid y \in Y\}$ is an clopen covering of Y . Since Y is a closed subset of the compact space X , it is itself compact. Thus there is a finite subcovering $\{V_{y_i} \mid i = 1, \dots, n\}$. The set $V = \bigcap_{i=1}^n V_{y_i}$ satisfies the conditions of the lemma. For (2) we let $y \in Y$. By part (1), for each $z \in Z$ there is a clopen set U_z such that $Y \subseteq U_z$ and $z \notin U_z$. The family $\{X \setminus U_z \mid z \in Z\}$ is a clopen covering of Z . Since Z is a closed subset of the compact space X , there is a finite subcovering $\{X \setminus U_{z_i} \mid z_i \in Z\}_{i=1}^n$. Let $X \setminus U = \bigcap_{i=1}^n U_{z_i}$. Then U satisfies the requirements of the lemma. \square

Example 2.5 It is convenient to have a few examples of an abstract Stone space. These examples are taken from [Dav90].

1. Any finite set with the discrete topology is trivially a Stone space.
2. The one point compactification of \mathbb{N} is a Stone space. The one point compactification of a topological space is formed by formally adjoining an additional point, usually denoted by ∞ (in our case we may call this set, say, \mathbb{N}_∞), and then defining the topology as follows: the open sets are the open sets in the original topology together with any set V containing ∞ such that the complement of V is closed and compact in the original topology (for more details the reader is referred to [Kel75]). Indeed, if $x \neq y$ in \mathbb{N}_∞ , then we may assume that $x \neq \infty$ and thus $\{x\}$ is a clopen set containing x but not y . Hence it is totally disconnected and compact.
3. The product space of a family of Stone spaces is again a Stone space.
4. A closed subset of a Stone space is again a Stone space.

2.3 Representation Results

Theorem 2.6 Let B be a boolean algebra. Then the map

$$\eta : B \longrightarrow B_{X_B}$$

given by $\eta(a) = X_a$ where $X_a := \{F \in X_B \mid a \in F\}$ for $a \in B$ is a boolean algebra isomorphism of B onto the boolean algebra of clopen subsets of the dual space X_B of B .

Proof. We first show that η is a lattice homomorphism.

We must show that $X_{a \vee b} = X_a \cup X_b$ and $X_{a \wedge b} = X_a \cap X_b$ for any $a, b \in B$. We showed in Lemma 2.2 that $X_{a \vee b} = X_a \cup X_b$. Now $X_{a \wedge b} = \{F \in X_B \mid a \in F \text{ and } b \in F\}$. But for filters, $a \wedge b \in F \Leftrightarrow a \in F \text{ and } b \in F$. Thus $X_{a \wedge b} = \{F \in X_B \mid a \in F \text{ and } b \in F\} = X_a \cap X_b$. Thus η is a lattice homomorphism. It is clearly surjective. To show that η is injective, let $a \neq b$. Then $a \not\leq b$ or $b \not\leq a$. Assume without loss of generality that $a \not\leq b$. Thus $a \wedge b' \neq 0$. Let $J = \uparrow (a \wedge b')$. Then J is a proper filter since $0 \notin J$. Thus we may apply corollary 1.16 to J so that we obtain a prime filter F , so that $J \subseteq F$. Thus $a \wedge b' \in F$, and hence $a \in F$ and $b' \in F$. Now, $b \notin F$ since otherwise $b \wedge b' = 0$ would be in F . But since prime filters are proper, this is impossible. Thus F is a prime filter containing a but not b . Hence $X_a \neq X_b$, showing that η is injective. We also have that $\eta(0) = X_0 = \emptyset$ since there are no prime filters

containing 0, because all prime filters are proper. Finally, since all filters are up-sets, $1 \in F$ for $F \in X_B$. Thus $X_1 = X_B$. As a final note we remember that we do not need to show explicitly that the negation is preserved, since it follows from the above. \square

Theorem 2.7 Let X be a Stone space and B_X the boolean algebra of clopen subsets of X . Let X_{B_X} be the dual space of B_X . Then the map

$$\varphi : X \longrightarrow X_{B_X}$$

given by $\varphi(y) = \{U \in B_X \mid y \in U\}$ for $y \in X$ is a homeomorphism.

Proof. Firstly, we must show that $\varphi(y)$ is a prime filter. Let $y \in X$, and let $U \in \varphi(y)$ and $U \leq V$, where $V \in B_X$. But V is a clopen subset of X and $U \subseteq V$. Hence $y \in V$, thus $V \in \varphi(y)$.

Let $U, V \in \varphi(y)$. Then $U \wedge V = U \cap V$, and obviously $y \in U \cap V$. Thus $U \wedge V \in \varphi(y)$. Let $U \vee V \in \varphi(y)$. Hence $U \cup V \in \varphi(y)$. Now $(y \in U \cup V) \Leftrightarrow (y \in U \text{ or } y \in V) \Leftrightarrow (U \in \varphi(y) \text{ or } V \in \varphi(y))$. Hence $\varphi(y)$ is a prime filter of B_X , and thus $\varphi(y) \in X_{B_X}$, for all $y \in X$.

Let $x \neq y$ in X . Since X is a Stone space, by Lemma 2.4, there exists a clopen set U such that $x \in U$ and $y \notin U$. Thus $U \in \varphi(x)$, but $U \notin \varphi(y)$. Thus $\varphi(x) \neq \varphi(y)$ showing that φ is one-to-one. To show that φ is continuous, it is sufficient by Lemma A.5 to show that the inverse images of basic open sets are open. For the remainder of the proof we will abuse notation somewhat and let X_A denote the collection of prime filters of B_X that contain A , even though A is a capital letter. This is done merely to emphasise that A is actually a clopen set here.

Thus, we must show that $\varphi^{-1}(X_A)$ is open for all $A \in B_X$. Indeed $\varphi(y) \in X_A$ iff $A \in \varphi(y)$ iff $y \in A$. Thus, $\varphi^{-1}(X_A) = A$, which is open. Thus φ is continuous.

Finally, by Lemma A.10, we simply need to show that φ is onto. Now, since $\varphi(X)$ is a compact subset of X_{B_X} it is closed. Suppose, by contradiction, that there is an $U \in X_{B_X} \setminus \varphi(X)$. Then, since X_{B_X} is a Stone space, by Lemma 2.4 there is a clopen subset V of X_{B_X} such that $\varphi(X) \cap V = \emptyset$. But, by Lemma 2.3 every clopen set is of the form X_A , for some $A \in B_X$. But since φ is one-to-one, we have $\varphi(X) \cap X_A = \emptyset$ implies $X \cap \varphi^{-1}(X_A) = \emptyset$. Thus $X \cap A = \emptyset$. Hence $A = \emptyset$, which is impossible, since $x \in X_A$. Thus φ is a homeomorphism. \square

As a final remark, we mention without proof that the above construction is functorial. That is, there is a dual equivalence between the category of boolean algebras and the category of Stone spaces.

Chapter 3 - Priestley Duality

3.1 Introduction

Having obtained all the needed duality theorems for boolean algebras, one may ask if it is possible to extend these results to bounded distributive lattices. Indeed, the pioneering work on this was done by Stone as well [Sto37]. In his paper, Stone used pure topological spaces (for the dual space of a distributive lattice) to obtain the representation (just as for boolean algebras). By a pure topological space we mean that, besides for the topology, there is no additional structure on the space (notably an order-theoretic structure). Properties like being compact and so on qualify as purely topological considerations. However, Hillary Priestley noticed that the representation could be simplified greatly if one endowed the topological space with a partial order [Pri70]. This should not come as a great surprise, since a larger amount of structure being present should decrease the effort in trying to reconstruct the lattice. We will eventually call the dual spaces of a distributive lattice a Priestley space and we will show that it has some properties analogous to Stone spaces in the case of boolean algebras. The definitions and results that follow have been taken from [Dav90].

3.2 Priestley Spaces

We will still take the set of all prime filters of a bounded distributive lattice L as the underlying set for the dual space (Recall the introductory paragraphs of Chapter 2 for the motivation for this). We are now faced with the same difficulty that we had in the boolean algebra case, that is, to show that the set of prime filters is non-empty. Fortunately for us, the lemmas which guarantee the existence of prime filters carry over from the previous section. We recall that in Lemma 1.15 we only required that the lattice be bounded and distributive, so we may again speak meaningfully about the set of prime filters. We must be careful here though. In the case of a boolean algebra, the prime filters are all maximal, but here that is no longer the case. We thus obtain a natural ordering on the set of prime filters: the inclusion order. We note that all maximal filters are still prime, even though there are some prime filters that are no longer maximal.

Unfortunately, there is another slight complication. We cannot, as before, use the family $\{X_a \mid a \in L\}$, where $X_a := \{F \in \mathcal{F}_p(L) \mid a \in F\}$ with L a bounded distributive lattice as the basis for our topology. This is because we no longer have the notion of a complement in this setting. If we just tried to apply the previous ideas without modifying them properly, we would obtain a space where each set X_a was not clopen. Clearly we would like to define

our mapping as in Theorem 2.6. Thus, in order to force each set X_a to be clopen, we need to consider the following. We will again abuse notation and let $X_L := \mathcal{F}_p(L)$, where L is a bounded distributive lattice. Then let \mathcal{S} be defined as,

$$\mathcal{S} := \{X_b \mid b \in L\} \cup \{X_L \setminus X_c \mid c \in L\}.$$

There is one final problem that we have to address. This set isn't closed under finite intersections. To solve this problem, let

$$\mathcal{B} := \{X_b \cap (X_L \setminus X_c) \mid b, c \in L\}.$$

Since L has 0 and 1, \mathcal{B} contains \mathcal{S} . \mathcal{B} is also closed under finite intersections by definition (Recall that $X_a \cup X_b = X_{a \vee b}$ and that $X_a \cap X_b = X_{a \wedge b}$). Thus we may use \mathcal{B} as a basis for a topology (with \mathcal{S} thus being a subbasis), and indeed, this is exactly what we do. Thus, let Ω_L be the topology on X_L , having \mathcal{B} as a basis. The task now before us is to investigate the properties of (X_L, Ω_L, \leq) .

Lemma 3.1 The dual space of a bounded distributive lattice is compact.

Proof. We shall apply Alexander's Subbasis Lemma. Let L be a bounded distributive lattice. Suppose thus that

$$\mathcal{U} = \{X_b \mid b \in B\} \cup \{X_L \setminus X_c \mid c \in C\}$$

is an open covering of X_L , for some subsets B and C of L for which there is no finite subcover. Thus for any finite subsets J and G with $J \subseteq B$ and $G \subseteq C$ we have $\bigcap \{X_c \mid c \in G\} \not\subseteq \bigcup \{X_b \mid b \in J\}$. Indeed if we had $\bigcap \{X_c \mid c \in G\} \subseteq \bigcup \{X_b \mid b \in J\}$ then we may union both sides with $X_L \setminus \bigcap \{X_c \mid c \in G\}$ thus obtaining $X_L \subseteq \bigcup \{X_b \mid b \in J\} \cup X_L \setminus \bigcap \{X_c \mid c \in G\}$, a contradiction to the fact that there is no finite subcover. Thus we have $\bigwedge G \not\leq \bigvee J$. This implies that the filter generated by C is disjoint from the ideal generated by B . By Lemma 1.15 there is a prime filter F such that $C \subseteq F$ and F also disjoint from the ideal generated by B . This means that $F \not\subseteq X_L \setminus X_c$ for any $c \in C$ and $F \not\subseteq X_b$ for any $b \in B$. This contradicts the fact that \mathcal{U} is a covering. Thus X_L is compact. \square

Definition 3.2 Let (X, \leq, Ω) be an ordered topological space. Then X is said to be totally order-disconnected iff given two points $x, y \in X$ with $x \not\leq y$, there exists a clopen up-set U such that $x \in U$ but $y \notin U$.

Lemma 3.3 The dual space X_L of a bounded distributive lattice L is totally order-disconnected.

Proof. Let $F \not\leq G$ with $F, G \in X_L$. Then there must be an $a \in F$ but $a \notin G$. We notice first that, because of the way we have chosen the subbasis for our open sets, each set X_a will be a clopen set. In order to prove that X is totally order-disconnected it will be sufficient to show that each X_a is also an up-set, since then X_a would be a clopen up-set containing F but not G . But X_a is trivially an up-set since if we let $U \in X_a$ and we let $V \in X_L$ and $U \leq V$. Then

$a \in V$. Thus $V \in X_a$. \square

From now on, we shall call compact totally order-disconnected spaces Priestley spaces. By combining the above two lemmas, we see that the dual space of a bounded distributive lattice is indeed a Priestley space.

Before we can prove the representation theorems, we shall need a few facts concerning the nature of clopen sets and clopen up-sets. We arrive next at another useful result concerning compact, totally order-disconnected spaces. This is the Priestley space analog of Lemma 2.4.

Corollary 3.4 Let X be a Priestley space.

1. Let Y be a closed up-set in X , and let $x \notin Y$. Then there is a clopen up-set U such that $Y \subseteq U$ and $x \notin U$.
2. Let Y and Z be disjoint closed subsets of X such that Y is a down-set and Z is an up-set. Then there exists a clopen up set U , $Z \subseteq U$ and $Y \cap U = \emptyset$.

Example 3.5 We pause very briefly to provide a few examples (these examples can be found in [Dav90]).

1. The collection of clopen up-sets of a compact, totally order-disconnected space is a bounded distributive lattice.
2. The natural numbers with the gcd (greatest common divisor) and lcm (lowest common multiple) operations representing the meet and join operations respectively is a distributive lattice with 0. Here $1 \in \mathbb{N}$ is the bottom element, since it is a divisor of every natural number.
3. The Lindenbaum algebra, which consists of equivalence classes of sentences of a particular logical theory is a bounded distributive lattice with the disjunction and conjunction acting as the lattice operations.

Lemma 3.6 Let L be a bounded distributive lattice and X_L its dual space. Then

1. The clopen sets are the finite unions of sets of the form $X_b \cap (X_L \setminus X_c)$ for $b, c \in L$.
2. The clopen up-sets are exactly the sets X_a .

Proof. For part 1, we let U be a clopen subset of X . Since U is open, it is a union of basic open sets. That is,

$$U = \bigcup_{a \in A, b \in B} (X_a \cap (X_L \setminus X_b))$$

for some $A, B \subseteq L$. But, since U is also closed, it is compact because X_L is compact. Thus, there is a finite subcovering. Hence

$$U = \bigcup_{a \in A_1, b \in B_1} (X_a \cap (X_L \setminus X_b))$$

where A_1 is a finite subset of A and B_1 is a finite subset of B . This is exactly the representation we were looking for. Also, every finite union of sets of the form $X_b \cap (X_L \setminus X_c)$ for $b, c \in L$ is trivially clopen.

For part 2, we note that each set X_a is clopen by part 1, and is an up-set since $F \in X_a$ with $F \subseteq G$, where G is any prime filter of L , implies that $a \in G$ and hence that $G \in X_a$. For the converse, note first that since X_a is an up-set, $X_L \setminus X_a$ is a down-set. Let V be any clopen up-set of X_L . By part 1, V is a finite union of sets of the form $X_b \cap (X_L \setminus X_c)$. Since V is an up-set, it must therefore be a finite union of sets of the form X_b . It then follows by the first part of Theorem 2.6 that $V = X_a$ for some $a \in L$. \square

Example 3.7 We give a few examples of Priestley spaces [Dav90] [Kel75].

1. Any set with the discrete topology is trivially a Priestley space.
2. The one point compactification of \mathbb{N} is a Priestley space. The natural ordering on \mathbb{N} is kept and we make the additional insistence that $n \leq \infty$ for any $n \in \mathbb{N}$. Indeed, if $x \not\leq y$ in \mathbb{N}_∞ , then we may assume that $x \neq \infty$ and thus $\uparrow x$ is a clopen up-set containing x but not y . Hence it is totally order-disconnected and compact. If $x = \infty$ then $\uparrow y \setminus \{y\}$ is the required up-closed set.

3.3 Representation Results

Theorem 3.8 Let L be a bounded distributive lattice. Then the map

$$\eta : L \rightarrow L_{X_L}$$

given by $\eta(a) = \{F \in X_L \mid a \in F\}$ for $a \in L$ is an isomorphism of L onto the lattice of clopen up-sets of the dual space of L .

Proof. We have already shown in the first part Theorem 2.6 that η is a lattice homomorphism and that it preserves 0 and 1. We merely have to show that η is one-to-one and that it is onto. Now, from Lemma 3.6 we know that the sets X_a are exactly all of the clopen up-sets, and for each X_a we may trivially pick $a \in L$ to map to it. Thus η is onto. Further, $a \leq b$ implies $X_a \subseteq X_b$ since filters are up-sets. Finally, to show the converse, suppose that $a \not\leq b$. Let $G = \uparrow a$ and $J = \downarrow b$. Then, by Lemma 1.15, there is a prime filter F containing G but which is disjoint from J . Thus $X_a \not\subseteq X_b$. Thus η is an order-embedding and hence one-to-one. \square

Theorem 3.9 Let X be a compact, totally order-disconnected space. Then the map

$$\varphi : X \rightarrow X_{L_X}$$

defined by $\varphi(y) := \{A \in L_X \mid y \in A\}$ for $y \in X$ is an order-homeomorphism of X onto the dual space X_{L_X} of the lattice of clopen up-sets L_X of X .

Proof. We showed in Theorem 2.7 that this causes $\varphi(y)$ to be a filter in L_X . Now, let $y \leq z$ in X and let $y \in A$ where $A \in L_X$. Then obviously $z \in A$ since A is an up-set. Since the choice of A was arbitrary, it follows by definition

that $\varphi(y) \leq \varphi(z)$. Conversely, suppose that $y \not\leq z$ in X . Since X is totally order-disconnected, there exists an $A \in L_X$ such that $y \in A$ but $z \notin A$. Thus $\varphi(y) \not\leq \varphi(z)$. Hence φ is an order-embedding and also one-to-one. To show that φ is continuous we will apply Lemma A.5. Here it suffices to show that $\varphi^{-1}(X_A)$ and $\varphi^{-1}(X_{L_X} \setminus X_A)$ are open for each $A \in L_X$. Since φ is one-to-one it follows that $\varphi^{-1}(X_{L_X} \setminus X_A) = \{y \in X \mid \varphi(y) \notin X_A\} = X \setminus \varphi^{-1}(X_A)$. Thus φ is continuous iff $\varphi^{-1}(X_A)$ is clopen in X for each $A \in L_X$. Now

$$\begin{aligned}\varphi^{-1}(X_A) &= \{y \in X \mid \varphi(y) \in X_A\} \\ &= \{y \in X \mid A \in \varphi(y)\} \\ &= A\end{aligned}$$

which is clopen, by the definition of L_X .

Finally we show that φ is onto since, by Lemma A.10, this will imply that φ is a homeomorphism. Now $\varphi(X)$ is a closed subset of X_{L_X} since it is compact. Suppose, by contradiction, that there is an $x \in X_{L_X} \setminus \varphi(X)$. Corollary 3.4 implies that there is a clopen up-set $-V$ of X_{L_X} such that $\varphi(X) \cap V = \emptyset$ and $x \in V$. Since $-V$ is clopen, so is V . By Lemma 3.6 we may assume that $V = X_B \cap (X_{L_X} \setminus X_C)$ for some $B, C \in L_X$. Since φ is one-to-one $\varphi(X) \cap V = \emptyset$ implies $\emptyset = X \cap \varphi^{-1}(V) = B \cap (X \setminus C)$. Hence $B \subseteq C$, which is impossible since $x \in X_B \cap (X_{L_X} \setminus X_C)$. Thus φ is an order-homeomorphism. \square

Chapter 4 - Dualities for Bounded Lattices

When we examined the results Priestley discovered regarding the representation of bounded distributive lattices, we mentioned briefly that Stone did the pioneering work on the topic, without delving into the details of his approach. We shall not be so lenient in the case of a general bounded lattice, which is discussed in the sections below. The reason for this is mainly that the aim of this thesis (as stated in the introduction) is to endow a general (not necessarily distributive) bounded lattice with structure-preserving maps, and then discover how to preserve this structure via the representation. To this end, we shall have to delve deeply into the inner workings of bounded lattice representations and we thus present here three approaches to the same problem, each with its own flavour and subtleties. In the end, we will choose the approach taken by Urquhart [Urq78]. The motivation for this is merely that we wished to extend the results of Goldblatt [Gol89] in the case of distributive lattices, and Goldblatt's work is based on Priestley duality, of which Urquhart's work is a natural generalization, as we shall see. For this reason, we present Urquhart's results first, as they will be most important to us. We then discuss the work of Hartonas [Har97] and finally the results of Moshier and Jipsen [Mos09].

4.1 Urquhart Duality

4.1.1 Introduction

The strategy taken by Urquhart is one in keeping with the approach taken by Priestley in [Pri70], in the sense that we associate an ordered topological space with a bounded lattice, instead of a 'pure' topological space. As stated in the introductory paragraph of this section, this will form the basis for our later work. Now, in order to understand what one needs to add on the topological side to make the duality go through, one needs to understand what it is that the distributivity accomplished in the first place: It provided a direct link between the joins and the meets. They could interact with each other via the distributivity axioms. The question then becomes: Which link should be separated on the topological side in order to mimick this non-distributivity correctly? The answer, as we shall see, turns out to be deceptively simple: simply turn the partial order (that we would have on a Priestley space) into two pre-orders, that is, orders that are reflexive and transitive but not anti-symmetric. Then, in order to ensure that the distributive case may be obtained as a specialization, we add one axiom: If two elements are related (one being less than the other) via both of the pre-orders, then the two elements must be equal. The distributive case may then be obtained by setting the one pre-order as the inverse of the other. The axiom that we have just described then turns

the two pre-orders into a partial ordering.

Now, in the distributive case, the order interacted with the topology in a fundamental way to give rise to the duality we needed: Indeed, we considered the clopen up-sets in that case. But how to proceed here?

The reader should note that, unless otherwise stated, all the definitions and results are due to Urquhart. In particular, the proof of Theorem 4.7 is the author's own version.

Let us define the mappings and structures that we will require.

(X, \leq_1, \leq_2) is called a doubly-ordered set if \leq_1 and \leq_2 doubly-order X and $x \leq_1 y$ and $x \leq_2 y$ imply $x = y$, where each of \leq_1 and \leq_2 are a pre-order.

$(X, \Omega, \leq_1, \leq_2)$ is called a doubly-ordered space if Ω is a topology on X and \leq_1 and \leq_2 doubly-order X .

Note that we will denote the set complement of Y relative to X , i.e. $X - Y := \{x \in X \mid x \notin Y\}$, by $-Y$ where X is understood. Define

$$\begin{aligned} l(Y) &= \{x \mid \forall y \in X, x \leq_1 y \Rightarrow y \notin Y\} \\ r(Y) &= \{x \mid \forall y \in X, x \leq_2 y \Rightarrow y \notin Y\}. \end{aligned}$$

These maps play a fundamental in understanding and establishing the duality for the bounded lattice case. A useful observation is that

$$\begin{aligned} l(Y) &= \{x \mid \uparrow_1 x \cap Y = \emptyset\} = -\downarrow_1 Y \text{ and} \\ r(Y) &= \{x \mid \uparrow_2 x \cap Y = \emptyset\} = -\downarrow_2 Y. \end{aligned}$$

The next three lemmas show some useful properties of l and r .

Lemma 4.1 Let (X, \leq_1, \leq_2) be a doubly ordered set. Then for any $A \subseteq X$, $l(A)$ is \leq_1 -increasing and $r(A)$ is \leq_2 -increasing.

Proof. Let $A \subseteq X$ and let $a \leq_1 b$, with $a \in l(A)$. Then, for any $d \in X$, $b \leq_1 d$ implies $a \leq_1 d$ which implies $d \notin A$. Hence $b \in l(A)$. The argument for $r(A)$ is symmetrical. \square

Lemma 4.2 Let (X, \leq_1, \leq_2) be a doubly-ordered set. Then for any $A, B \subseteq X$,

1. l and r are antitone maps. That is, $A \subseteq B$ implies $l(B) \subseteq l(A)$ and $r(B) \subseteq r(A)$,
2. $r(A \cup B) = r(A) \cap r(B)$ and $l(A \cup B) = l(A) \cap l(B)$.
3. If A is \leq_2 -increasing, then $A \subseteq rl(A)$. Similarly, if B is a \leq_1 -increasing set, then $B \subseteq lr(B)$.

Proof. For (1), if $A \subseteq B$, then $\downarrow_1 A \subseteq \downarrow_1 B$, which implies that $-\downarrow_1 B \subseteq -\downarrow_1 A$ showing that $l(B) \subseteq l(A)$. The case $r(B) \subseteq r(A)$ is similar. For (2), $r(A \cup B) = -\downarrow_2 (A \cup B) = -(\downarrow_2 A \cup \downarrow_2 B) = -\downarrow_2 A \cap -\downarrow_2 B = r(A) \cap r(B)$.

The case $l(A \cup B) = l(A) \cap l(B)$ is similar. For (3), let $a \in A$ and let $a \leq_2 b$. We want $b \notin l(A)$. Now because A is \leq_2 -increasing, $b \in A$. But $b \leq_1 b$ by the reflexivity of \leq_1 . Hence $b \notin l(A)$. \square

An important fact concerning the maps l and r is given in the following lemma.

Lemma 4.3 Let (X, \leq_1, \leq_2) be a doubly-ordered set. Then the mappings l and r , defined as above, form a Galois connection between the lattice of \leq_1 -increasing- and the lattice of \leq_2 -increasing subsets of X . Moreover, the composition lr is a closure operator on the lattice of \leq_1 -increasing subsets of X . Similarly, rl is a closure operator on the lattice of \leq_2 -increasing subsets of X .

Proof. We first show that lr is a closure operator on \leq_1 -increasing sets. For any \leq_1 -increasing sets A, B ,

1. $x \in A$ and $x \leq_1 y$ imply $y \in A$. So, by reflexivity of \leq_2 , there is some $u \in X$, namely $u = y$ such that $y \leq_2 u$ and $u \in A$. Thus $y \notin r(A)$, showing that $A \subseteq lr(A)$.
2. lr is monotone since l and r are antitone.
3. From (1) and by monotonicity of lr , $lr(A) \subseteq lr(lr(A))$. On the other hand, assume $x \in lr(lr(A))$ and $x \leq_1 y$. Then $y \notin rlr(A)$. But since $r(A)$ is \leq_2 -decreasing, $rA \subseteq rl(r(A))$ and hence $y \notin rA$. Thus $x \in lr(A)$ as required.

Finally, l and r form a Galois connection, that is, for any \leq_2 -increasing set A and any \leq_1 -increasing set B ,

$$B \subseteq l(A) \quad \text{iff} \quad A \subseteq r(B)$$

Indeed, assume $B \subseteq l(A)$. Then, since r is antitone, $rl(A) \subseteq r(B)$. Thus, since A is \leq_2 -increasing, $A \subseteq rl(A) \subseteq r(B)$. Assume $A \subseteq r(B)$. Then, since l is antitone, $lr(B) \subseteq l(A)$. Thus, since B is \leq_1 -increasing, $B \subseteq lr(B) \subseteq l(A)$. \square

A subset Y of a doubly-ordered set X is called l -stable (resp. r -stable) if $Y = lr(Y)$ (resp. $Y = rl(Y)$). Since we shall be referring to l -stable sets for most of the remaining part of this section, we shall simply call l -stable sets stable sets. The following properties of stable sets will be needed later. For a more elaborate list of properties of the maps l and r , we refer the reader to [Rew05].

Lemma 4.4 If Y, Z are stable subsets of a doubly-ordered topological space X , then

1. $r(Y) \subseteq -Z \Rightarrow r(Y) \subseteq r(Z)$
2. $Y \subseteq Z, r(Y) \subseteq r(Z) \Rightarrow Y = Z$
3. $Y \subseteq -r(Z) \Rightarrow Y \subseteq Z$.

Proof. For (1), let $r(Y) \subseteq -Z$ and let $x \in r(Y)$. We want $x \in r(Z)$. Take any $d \in X$ with $x \leq_2 d$. Then, since $r(Y)$ is \leq_2 -increasing, $d \in r(Y)$. Hence $d \notin Z$, as required. For (2), we simply apply l to both sides of $r(Y) \subseteq r(Z)$. For (3) we notice that

$$\begin{aligned} Y \subseteq -r(Z) &\Rightarrow r(Z) \subseteq -Y && \text{(property of the complement)} \\ &\Rightarrow r(Z) \subseteq r(Y) && \text{by (1)} \\ &\Rightarrow Y \subseteq Z && \text{by (2)} \end{aligned}$$

□

We now elaborate about exactly which conditions we want our closed sets to satisfy. When the context is clear, we shall refer to the doubly-ordered space under consideration merely as X , in keeping with the philosophy outlined in Section 1.1.

Definition 4.5 A subset Y of a doubly-ordered space X is called doubly-closed if both Y and $r(Y)$ are closed in X .

Definition 4.6 Let X be a doubly-ordered topological space. Then X is defined to be doubly-disconnected if

1. for any $x, y \in X$, $x \not\leq_1 y$ implies there exists a doubly-closed stable set Y such that $x \in Y$ but $y \notin Y$.
2. for any $x, y \in X$, $x \not\leq_2 y$ implies there exists a doubly-closed stable set Y such that $x \in r(Y)$ but $y \notin r(Y)$.

That concludes all the definitions that we shall require. It is time to describe the representation. Let us try to understand going from a bounded lattice to an Urquhart space first. We essentially have to answer three questions (keeping in mind that we are generalizing Priestley's work):

1. What happens to the partial ordering when distributivity is lost?
2. What will the new underlying set for the dual space be?
3. How will the open sets be defined?

We answered question 1 in the opening paragraph of this section. For question 2, we should not choose recklessly, lest we lose the ability to specialize to the distributive case. Recall that, in that case, we considered the set of prime filters of a distributive lattice. The ordering on the topological space was then inherited from the inclusion ordering that existed on the prime filters. As we saw, we shall be dealing with spaces with two orderings on them. This suggests that we should choose our points for the underlying set in such a way that both of our pre-orders may be derived from inclusion orders. If we then remember that we want the one pre-order to be the inverse of the other during specialization to the distributive case, we are compelled to consider both filters and ideals of our lattice. This is because ideals and filters are essentially order-dual to each other, that is, inverting the order in a lattice turns ideals to filters and vice versa. It appears then that our house will be in order if we choose

points to be filter-ideal pairs $\langle F, I \rangle$, where F is a filter- and I is an ideal of our bounded lattice and we define $\langle F_1, I_1 \rangle \leq_1 \langle F_2, I_2 \rangle$ iff $F_1 \subseteq F_2$ and similarly $\langle F_1, I_1 \rangle \leq_2 \langle F_2, I_2 \rangle$ iff $I_1 \subseteq I_2$. This is a good first step, but it is not sufficient because of the extra axiom that we insisted upon: In our spaces, $x \leq_1 y$ and $x \leq_2 y$ should imply that $x = y$. Fortunately, we can remedy the matter by modifying the pairs in the following way. We insist firstly that they should be disjoint, and secondly that F should be maximal in the family of all filters disjoint from I , and that I should be maximal in the family of all ideals disjoint from F . Let us call filter-ideal pairs satisfying these conditions maximal pairs.

This solves the problem because of the following argument: If $\langle F_1, I_1 \rangle \leq_1 \langle F_2, I_2 \rangle$, then (in addition to the definition) we may deduce that $I_2 \subseteq I_1$. This is so since $F_1 \subseteq F_2$ implies that F_1 is also disjoint from I_2 . But I_1 is maximal in the family of ideals disjoint from F_1 . It then follows that, for maximal pairs $\langle F_1, I_1 \rangle$ and $\langle F_2, I_2 \rangle$, $\langle F_1, I_1 \rangle \leq_1 \langle F_2, I_2 \rangle$ and $\langle F_1, I_1 \rangle \leq_2 \langle F_2, I_2 \rangle$ implies $\langle F_1, I_1 \rangle = \langle F_2, I_2 \rangle$. So, in summary, we take for the underlying set of the dual space the collection of all maximal pairs. Now, as was the case for distributive lattices and prime filters, we need to establish analogously that the collection of maximal pairs is non-empty. This motivates the following lemma.

Lemma 4.7 If $\langle F, I \rangle$ is a pair in L , then $F \subseteq x_1$, $I \subseteq x_2$ for some maximal pair $x = \langle x_1, x_2 \rangle$ in L .

Proof. Let L be a bounded lattice and F and I a filter and ideal respectively such that they have empty intersection. Define $\mathcal{S} = \{K \mid K \supseteq F \text{ and } K \cap I = \emptyset \text{ where } K \text{ is a filter.}\}$. The set contains F , and so is non-empty. Let $C = \{U_i\}$ be a chain in \mathcal{S} . To apply Lemma 1.14, we require that $U := \bigcup_i U_i \in \mathcal{S}$.

Thus we must show that U is a filter containing F and that it is disjoint from I . Since each U_i contains F , it follows that U also contains F . Further, because each U_i is up closed - all of them being filters - we have that U is up-closed. Also, if $x \in U \cap I$, then $x \in U_i \cap I$ for some i , which is impossible. Hence U is disjoint from I . It remains to show that $x, y \in U \Rightarrow x \wedge y \in U$. Now $x \in U \Rightarrow \exists i_1$ such that $x \in U_{i_1}$. Similarly, $y \in U_{i_2}$ for some i_2 . But C is a chain. Assume therefore without loss of generality that $U_{i_1} \subseteq U_{i_2}$. Thus $x, y \in U_{i_2}$. Hence $x \wedge y \in U_{i_2} \subseteq U$. Thus by Lemma 1.14 the collection \mathcal{S} has a maximal element F^* .

Now we apply the exact same argument to the pair $\langle F^*, I \rangle$. Define $\mathcal{S} = \{K \mid K \supseteq I \text{ and } K \cap F^* = \emptyset \text{ where } K \text{ is an ideal.}\}$. The set contains I , and so is non-empty. Let $C = \{V_i\}$ be a chain in \mathcal{S} . To apply Lemma 1.14, we require that $V := \bigcup_i V_i \in \mathcal{S}$.

Thus we must show that V is an ideal containing I and that it is disjoint from F^* . Since each V_i contains I , it follows that V also contains I . Further, because each V_i is down closed - all of them being ideals - we have that V is down-closed. Also, if $x \in V \cap F^*$, then $x \in V_i \cap F^*$ for some i , which is impossible. Hence V is disjoint from U . It remains to show that $x, y \in V \Rightarrow x \vee y \in V$. Now $x \in V \Rightarrow \exists i_1$ such that $x \in V_{i_1}$. Similarly, $y \in V_{i_2}$ for some i_2 . But C is a chain. Assume therefore without loss of generality that $V_{i_1} \subseteq V_{i_2}$. Thus $x, y \in V_{i_2}$. Hence $x \vee y \in V_{i_2} \subseteq V$. Thus by Lemma 1.14 the collection \mathcal{S} has a maximal element I^* .

The pair $\langle F^*, I^* \rangle$ satisfies the conditions of the lemma. \square

Let us make use of our notational convention and denote the collection of all maximal pairs of a bounded lattice L by X_L . Our arguments so far have made X_L into a doubly-ordered set. As a last remark before we move on to the next question, note that, in the distributive case, where we consider prime filters, the filter- and the ideal parts of the maximal pair will be set complements of each other, so that the set of maximal pairs is equivalent to the set of prime filters in that case since the ideal part is redundant.

We now set out to answer question 3. The guiding light is once more the distributive case. Recall that the clopen up-sets were exactly those collections of prime filters which contained specific elements of the lattice L . However, the points in our space have two ‘components’, a filter part and an ideal part. So we consider the following two collections for any $a \in L$:

$$F_a = \{x \in X_L \mid a \in x_1\},$$

$$I_a = \{x \in X_L \mid a \in x_2\}.$$

So where do the two pre-orders enter into the picture here? We shall prove shortly that the following two equalities hold:

$$r(F_a) = I_a, \tag{4.1}$$

$$l(I_a) = F_a. \tag{4.2}$$

Recall also the earlier result which stated that the composition of l and r are closure operators. Should we not then also give credence to those sets which are closed with respect to these closure operators? Moreover, how are these sets related to the collections F_a and I_a defined above? The following lemma elaborates on these questions.

Lemma 4.8 Let L be a bounded lattice. Then for any $a \in L$,

1. $r(F_a) = \{x \in X_L \mid a \in x_2\}$,
2. F_a is a stable set in X_L .

Proof. Let $x \in X_L$ with $a \in x_2$. Let $x \leq_2 y$. Then $a \in y_2$ and thus $a \notin y_1$. Hence $y \notin F_a$. Thus, by definition, $x \in r(F_a)$. Conversely, if $x \notin \{x \in X_L \mid a \in x_2\}$, then $a \notin x_2$. Thus $\uparrow a \cap x_2 = \emptyset$ and hence by Lemma 4.7 there is a maximal pair $y = \langle y_1, y_2 \rangle$ such that $\uparrow a \subseteq y_1$ and $x_2 \subseteq y_2$. Hence $x \leq_2 y$ but $y \notin F_a$. Thus $x \notin r(F_a)$. For (2), we require that $lr(F_a) = F_a = \{x \in X_L \mid a \in x_1\}$. Let $x \in X_L$ with $a \in x_1$. Let $x \leq_1 y$ where $y \in X$ we want $y \notin r(F_a)$. Now, $x \leq_1 y$ implies that $a \in y_1$, thus $a \notin y_2$. Hence, by part (1), $y \notin r(F_a)$. Thus $x \in lr(F_a)$. Conversely, if $x \notin \{x \in X \mid a \in x_1\}$, then $a \notin x_1$. Thus $x_1 \cap \downarrow a = \emptyset$ thus creating a filter-ideal pair. By Lemma 4.7 there is a maximal pair y such that $x_1 \subseteq y_1$ and $\downarrow a \subseteq y_2$. Thus $x \leq_1 y$ but $y \in r(F_a)$ by part (1). Hence $x \notin lr(F_a)$. Hence F_a is stable. \square

Notice that Lemma 4.8 implies Equation 4.2 since $lI_a = lrF_a = F_a$. Moreover, we are almost vindicated in our decision to consider the collections F_a

and I_a . Indeed, Lemma 4.8 shows that, for any $a \in L$, these sets are closed with respect to the closure operators lr and rl respectively. We cannot allow the dual space of a bounded lattice to be endowed with two distinct (albeit highly related) closure systems however. Nor can we afford to throw any of the two closure systems away. We therefore consider as a candidate for our topology on X_L the smallest topology for which both the l -stable sets and the r -stable sets are closed. That is, we endow X_L with the topology generated by having the family $\{-F_a\} \cup \{-I_b\}$ as a subbasis, where a, b range over L . We may denote this topology by Ω_L . This turns X_L into a doubly-ordered space, called the dual space of L .

Note that, for any $a \in L$, F_a is a doubly-closed stable set. Because of the close connection between the sets F_a and I_a via the maps l and r , we will make the choice of referring to I_a as rF_a whenever possible, since this way we only have to worry about one of them. The choice is arbitrary and, by symmetry, doesn't affect our future results. We may now ponder the converse: Are all doubly-closed stable sets of the form F_a ? We answer this question in the affirmative shortly. This takes care of the issue of constructing a doubly-ordered space given a bounded lattice. We saw that Priestley spaces have some interesting topological properties that were important in making the duality go through. Recall that a Priestley space is a compact, totally order-disconnected space. Definition 4.6 is the doubly-ordered space analog of being totally order-disconnected. It seems inevitable that we shall need the dual space to be compact in the bounded lattice case as well.

4.1.2 Urquhart Spaces

Having observed all of the above, we are cautiously optimistic and make the following definition.

Definition 4.9 Let X be a doubly-ordered space. X is called an Urquhart space if

1. X is doubly-disconnected and compact.
2. Y, Z doubly-closed stable sets in X imply that $r(Y \cap Z)$ and $l(r(Y) \cap r(Z))$ are closed in X .
3. The family $\{-Y \mid Y \text{ a doubly-closed stable set}\} \cup \{-r(Y) \mid Y \text{ a doubly-closed stable set}\}$ forms a subbase for X .

An example of an Urquhart space can be found at the end of this section. The next lemma shows that constructing a bounded lattice given an Urquhart space is an altogether less involved process.

Lemma 4.10 If X is an Urquhart space, then the family of all doubly-closed stable sets forms a bounded lattice with $0 = \emptyset$, $1 = X$ and where the lattice ordering is inclusion and the lattice operations are given by the equations:

$$\begin{aligned} Y \wedge Z &= Y \cap Z \\ Y \vee Z &= l(r(Y) \cap r(Z)). \end{aligned}$$

Proof. Assume Y, Z are doubly-closed stable sets. Then Y, Z are \leq_1 -increasing and hence $Y \cap Z, Y \cup Z$ are \leq_1 -increasing. For clarity, the proof is broken into four parts.

1. $Y \cap Z$ is stable.

Since $Y \cap Z$ is \leq_1 -increasing, $Y \cap Z \subseteq lr(Y \cap Z)$. On the other hand, since $Y \cap Z \subseteq Y$ and r, l are anti-tone, $lr(Y \cap Z) \subseteq lrY = Y$. Similarly, $lr(Y \cap Z) \subseteq Z$. Hence, by definition of the greatest lower bound, $lr(Y \cap Z) \subseteq Y \cap Z$ as required.

2. $Y \vee Z = lr(Y \cup Z)$ is stable.

Since $Y \cup Z$ is \leq_1 -increasing, $lr(lr(Y \cup Z)) = lr(Y \cup Z)$, that is, $lr(Y \vee Z) = Y \vee Z$.

3. $Y \cap Z$ is doubly-closed.

Since Y and Z are closed in X , $Y \cap Z$ is closed in X . Since Y and Z are doubly-closed and stable, by Def 4.9 (2), $r(Y \cap Z)$ is closed in X .

4. $Y \vee Z$ is doubly-closed, that is $lr(Y \cup Z)$ and $r(lr(Y \cup Z))$ are closed in X . Since Y and Z are double-closed and stable, by Definition 4.9 (2), $lr(Y \cup Z)$ is closed in X . Since Y, Z are stable ($rY \cap rZ = rl(rY \cap rZ)$). [See properties of l -stable and r -stable sets.] Also, since rY and rZ are closed, $rY \cap rZ$ is closed in X and hence $r(lr(Y \cup Z))$ is closed in X .

□

We finally prove in the lemmas that follow that the dual space of a bounded lattice is indeed an Urquhart space. We make one last useful definition before we commence.

Definition 4.11 A family \mathcal{U} of doubly-closed stable sets in an Urquhart space X is said to be a separating family if for any $x, y \in X$ with $x \not\leq_1 y$ there exists $Z \in \mathcal{U}$, $x \in Z$ but $y \notin Z$.

Lemma 4.12 If X is a compact doubly-ordered space and \mathcal{U} is a separating family containing the set X and \emptyset and closed under \vee and \wedge then \mathcal{U} is the family of all doubly-closed stable sets in X .

Proof. Let W be an arbitrary doubly-closed stable set in X , and x an arbitrary point in W , i.e. $x \in lr(W)$. Thus $x \leq_1 y \Rightarrow y \notin r(W)$. Thus for $y \in r(W)$, $x \not\leq_1 y$ so that $x \in Z_{xy}$, $y \notin Z_{xy}$ for some $Z_{xy} \in \mathcal{U}$. That is, for each element $y \in r(W)$ we find a member Z_{xy} of the separating family such that $y \in -Z_{xy}$. Hence $r(W) \subseteq \bigcup \{-Z_{xy} \mid y \in r(W)\}$. This forms an open covering. By compactness, $r(W) \subseteq -Z_{xy_1} \cup \dots \cup -Z_{xy_m}$ for some m . Thus $r(W) \subseteq -(Z_{xy_1} \cap \dots \cap Z_{xy_m}) = -(Z_{xy_1} \wedge \dots \wedge Z_{xy_m})$. Let $Z_x = Z_{xy_1} \wedge \dots \wedge Z_{xy_m}$. Hence, by Lemma 4.4, $r(W) \subseteq r(Z_x)$. Since $x \in Z_x$ for any x , $W \subseteq \bigcup \{-r(Z_x) \mid x \in W\}$. This is an open covering of W . By compactness, $W \subseteq -r(Z_{x_1}) \cup \dots \cup -r(Z_{x_n})$ for some n . That is, $W \subseteq -(r(Z_{x_1}) \cap \dots \cap r(Z_{x_n})) = -r(Z_{x_1} \cup \dots \cup Z_{x_n})$. Let $Z = Z_{x_1} \vee \dots \vee Z_{x_n}$. Thus we have $W \subseteq -r(Z)$. By Lemma 4.4, $W \subseteq Z$. But since $r(W) \subseteq r(Z)$ it follows again by Lemma 4.4 that $W = Z$. Thus W is in \mathcal{U} . □

Lemma 4.13 If L is a bounded lattice, then X_L is an Urquhart space.

Proof. We again call upon Alexander's Subbasis lemma to show compactness. We argue by contradiction. Let \mathcal{U} be a subfamily of the subbase for

X_L such that no finite subfamily of \mathcal{U} covers X , but $\bigcup \mathcal{U}$ is an open covering. Define $U_1 = \{a \mid -F_a \in \mathcal{U}\}$, $U_2 = \{b \mid -r(u(b)) \in \mathcal{U}\}$. For any $\{a_1, \dots, a_m\} \subseteq U_1$, $\{b_1, \dots, b_n\} \subseteq U_2$, there is some $x \in X$ such that $x \in u(a_1) \cap \dots \cap u(a_m) \cap r(u(b_1)) \cap \dots \cap r(u(b_n))$, since no finite subfamily covers X (there is thus something left in the complement of any such finite subfamily). Thus $a_1 \wedge \dots \wedge a_m \in x_1$, $b_1 \vee \dots \vee b_n \in x_2$. Since x_1 and x_2 are disjoint, $a_1 \wedge \dots \wedge a_m \not\leq b_1 \vee \dots \vee b_n$. Hence the filter F_0 generated by U_1 is disjoint from the ideal I_0 generated by U_2 . Hence, by Lemma 4.7, there is a maximal pair $\langle y_1, y_2 \rangle$ such that $F_0 \subseteq F$ and $I_0 \subseteq I$. For $a \in U_1$, $y \in F_a$ and for $b \in U_2$, $y \in r(u(b))$ hence $y \notin \bigcup \mathcal{U}$, a contradiction. Thus X_L is compact. Lemma 4.8 implies that X_L is doubly-disconnected. The family $\mathcal{F} = \{F_a \mid a \in L\}$ satisfies the conditions of Lemma 4.12 (remember that $u(0) = \emptyset$ and $u(1) = X_L$) if we recall that the separating condition follows from the fact that the space is doubly-disconnected. Thus condition (2) of Definition 4.9 is met since \mathcal{F} is closed under \wedge . Thus X_L is an Urquhart space. \square

4.1.3 Representation Results

At last we are able to prove the two representation theorems.

Theorem 4.14 Let L be a bounded lattice. Then the map

$$f : L \rightarrow L_{X_L}$$

defined by $a \mapsto F_a := \{x \in X_L \mid a \in x_1\}$ is an isomorphism of L onto L_{X_L} .

Proof. Firstly, f is onto. Indeed, since all doubly-closed stable sets of X_L are of the form F_a for some $a \in L$, we may trivially map a to F_a . Further, f is an order-embedding. Indeed, let $a \leq b$ with $a, b \in L$ and let $x \in F_a$. Then $a \in x_1$. But since x_1 is a filter, and hence up-closed, $b \in x_1$. Thus $x_1 \in F_b$. Hence $F_a \subseteq F_b$. If $a \not\leq b$ then $\uparrow a \cap \downarrow b = \emptyset$ and thus, by Lemma 4.7 $a \in x_1$, $b \in x_2$ for some $x \in X$. Thus $F_a \not\subseteq F_b$. Lastly, by Lemma 4.12 and Lemma 4.13, the family $\{F_a\}$ is closed under meets and joins, thus $F_{a \vee b} = F_a \vee F_b$ and $F_{a \wedge b} = F_a \wedge F_b$. Putting all these facts together proves the theorem. \square

Theorem 4.15 If X is an Urquhart space then the map

$$v : X \rightarrow X_{L_X}$$

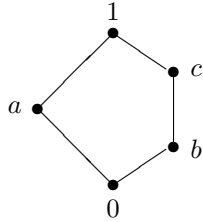
defined by $x \mapsto v(x) := \langle v_1, v_2 \rangle = \langle \{Y \in L(S) \mid x \in Y\}, \{Y \in L(S) \mid x \in r(Y)\} \rangle$ is an order-homeomorphism of X onto X_{L_X} .

Proof. Let X be an Urquhart space, and define the mapping v as above. It is clear that v_1 is a filter in L_X , but we prove that v_2 is an ideal. Indeed, it is trivially a down-set. Let $Y, Z \in L_X$. Then by Lemma 4.2, $r(Y) \cap r(Z) = r(Y \cup Z) \subseteq rlr(Y \cup Z) = rl(r(Y) \cap r(Z)) = r(Y \vee Z)$. Thus $x \in r(Y \vee Z)$, and hence $Y \vee Z \in L_X$. Thus v_2 is an ideal. We show that $v(x)$ is maximal. If F is a filter in L_X properly containing v_1 then $\bigcap F \subseteq \bigcap v_1(x)$, $x \notin \bigcap F$. Now if z is any element of $\bigcap F$ then $x \leq_1 z$ for otherwise $x \in Y$, $z \notin Y$ for some $Y \in L_X$, contradicting $z \in \bigcap v_1(x)$. Because $z \neq x$, it

follows that $x \not\leq_2 z$ so $x \in r(Y_z)$, $z \notin r(Y_z)$ for some $Y_z \in L_X$. Since $\bigcap F \subseteq \bigcup \{-r(Y_z) \mid z \in \bigcap F\}$, by compactness, $\bigcap F \subseteq -r(Y_{z_1}) \cup \dots \cup -r(Y_{z_m})$ for some m . Letting $Y = Y_{z_1} \vee \dots \vee Y_{z_m}$ we have $r(Y) \subseteq \bigcup \{-W \mid W \in F\}$. Hence by compactness, $r(Y) \subseteq -W_1 \cup \dots \cup -W_n$ for some W_1, \dots, W_n in F . Since $W_1 \wedge \dots \wedge W_n \subseteq -r(Y)$, $Y \in F$ by Lemma 4.4. By construction, $Y \in v_2(x)$ so that $v_1(x)$ is $v_2(x)$ -maximal. Similarly, $v_2(x)$ is $v_1(x)$ -maximal so that $v(x)$ is a maximal pair. To show that v is onto, let $\langle F, I \rangle$ be a maximal pair in L_X and let $F^* = F \cup \{r(Z) \mid Z \in I\}$. F^* has the finite intersection property, for if not then $Y \cap r(Z) = \emptyset$ for some $Y \in F$, $Z \in I$ so that $Y \subseteq -r(Z)$, hence $Y \subseteq Z$, a contradiction. Thus F can be extended to a maximal filter which by compactness has an adherent point x , hence $F \subseteq v_1(x)$, $I \subseteq v_2(x)$ so $v(x) = \langle F, I \rangle$. It is clear that v is an isomorphism with respect to both \leq_1 and \leq_2 so that v is one-to-one since if $x \neq y$ then either $x \not\leq_1 y$ or $x \not\leq_2 y$ so $v(x) \neq v(y)$. Finally, the function $Y \mapsto v^{-1}(Y)$ maps the subbase for X_{L_X} onto the subbase for X so that v is a homeomorphism. \square

In his paper, Urquhart shows that the above results are in fact functorial. As usual, we do not examine the details of this. We end off this section with an example which aims to clarify the process by which the above results operate.

Example 4.16 Consider the following bounded lattice:



Denote the lattice by L .

We will calculate in full the Urquhart duality for this lattice, showing explicitly how the dual space and dual lattice of the dual space is constructed.

We calculate the prime filters and ideals of L first and denote them as indicated below.

Filters

- $\{1\} \longrightarrow F_1$
- $\{a, 1\} \longrightarrow F_2$
- $\{c, 1\} \longrightarrow F_3$
- $\{b, c, 1\} \longrightarrow F_4$

Similarly for the ideals we have the following:

Ideals

- $\{0\} \longrightarrow I_1$
- $\{0, a\} \longrightarrow I_2$
- $\{0, b\} \longrightarrow I_3$
- $\{0, b, c\} \longrightarrow I_4$

This leads to the following set of maximal pairs:

Maximal Pairs

- $\langle F_4, I_2 \rangle \longrightarrow x$
- $\langle F_2, I_4 \rangle \longrightarrow y$
- $\langle F_3, I_3 \rangle \longrightarrow z$

This set of course forms the underlying set X_L for the dual space of L . Next we have to calculate the l -stable and r -stable subsets. In order to do this, we need the two pre-orders on X_L that the maximal pairs define. We tabulate those facts below.

$x \leq_1 x$	$x \leq_2 x$
$y \leq_1 y$	$y \leq_2 y$
$z \leq_1 z$	$z \leq_2 z$
$z \leq_1 x$	$z \leq_2 y$

In the next table we calculate the stability of the various subsets of X_L . Of course a set U is l -stable if $U = lrU$ and r -stable if $U = rlU$.

U	lrU	rlU
\emptyset	\emptyset	\emptyset
$\{x\}$	$\{x\}$	$\{x\}$
$\{y\}$	$\{y\}$	$\{y\}$
$\{z\}$	\emptyset	\emptyset
$\{x, y\}$	X	X
$\{x, z\}$	$\{x, z\}$	$\{x\}$
$\{y, z\}$	$\{y\}$	$\{y, z\}$
X	X	X

Now according to Urquhart's result, the lattice L should be isomorphic to the lattice of doubly-closed stable subsets of the dual space X_L of L . Let us therefore look at what the maps $F_{(.)}$ and $rF_{(.)}$ do to the various elements of L .

	$F_{(.)}$	$rF_{(.)}$
0	\emptyset	X
a	$\{y\}$	$\{x\}$
b	$\{x\}$	$\{y, z\}$
c	$\{x, z\}$	$\{y\}$
1	X	\emptyset

We can see clearly from the above calculations that the elements of the lattice L under the mapping $F_{(.)}$ correspond exactly to the left-stable sets and similarly that the right-stable sets correspond exactly to the elements of the lattice under the mapping $rF_{(.)}$. Furthermore we have seen that there are exactly the same amount of left- and right-stable sets because lY is left-stable for any right-stable set Y and similarly that rY is always right-stable for any left-stable set Y . Furthermore, it can readily be seen that the topology on X_L will be discrete, showing that the left-stable sets are indeed doubly-closed as we would expect from Urquhart's result. Here we can see very clearly how the topology is sewn together from the two closure operators lr and rl . Finally, under the relation of containment we see that here L is indeed isomorphic to L_{X_L} .

4.2 Hartonas Duality

4.2.1 Introduction

We have seen in the previous section how Urquhart used maximal filter-ideal pairs in order to construct the dual space of a bounded lattice. In his paper, Hartonas [Har97] uses a slightly different approach. He instead uses all the filters of a lattice as the underlying set of the dual space. This seems to be a very natural way of doing things - given the success of maximal filters in the boolean case and prime filters in the distributive case. The easiest way to understand the motivation for this choice is to recall the idea that Stone originally had. We repeat that idea here without the reader having to page back. Model an element b of a boolean algebra B by the set of homomorphisms $h : B \rightarrow 2$ (where 2 is the two-element boolean algebra) such that $h(b) = 1$. These homomorphisms are determined by the inverse image $h^{-1}(1)$ and moreover, $h^{-1}(1)$ is an ultrafilter of B .

However, we run into a small problem when we try to apply this idea to the case of an arbitrary bounded lattice. The inverse image $h^{-1}(1)$ in a general lattice is still a prime filter. Here is where Hartonas steps in to provide the answer: Instead of taking general lattice homomorphisms, he proposes that we take two hemimorphisms (one for the join and one for the meet). As the name suggests, a hemimorphism preserves ‘half’ of the structure of a lattice. The reason for doing this is that, in the case of a hemimorphism, the inverse image of 1 is indeed merely a filter. It seems then that each of the hemimorphisms ‘forgets’ the structural part of the lattice that it does not itself preserve. This, in essence, splits up the lattice into two semilattices. Hartonas’ plan is then to ‘glue’ together the two semilattices on which the hemimorphisms were defined (of course these notions will be made precise). This idea works because any lattice can be viewed as a combination of two meet-semilattices (the joins being an inverted version of the meets). Hartonas then says the following: ‘Our intuition has then been that it should be possible to reduce the lattice representation problem to that for (meet) semilattices as long as we understand the essence of this ‘glueing’ together’. So this finally motivates the choice for choosing filters as the underlying set of the dual space.

Of course, any representation result needs to do the opposite. That is, construct a lattice given an appropriate topological space. Here too, Hartonas notices a pattern. In the case of boolean algebras and distributive lattices, we take subsets of a closure system on the set underlying the dual space. In the boolean case, it was the full topological closure. In the distributive case we considered the clopen up-sets. These two operators have a very important property: they distribute over joins, making the resulting lattices distributive. How can we sidestep this problem? That is, how can we define a clever closure system on our topological space so that the closure operator does not distribute over union? Again Hartonas has the answer. All we need is an ordered topological space. To see this, we observe that we may create a closure operator from two maps forming a Galois connection by composing the them. Galois connections may in turn be found from a binary relation, as illustrated by the following lemma, which is stated without proof.

Unless otherwise stated, all the definitions and results that follow are due to Hartonas.

Lemma 4.17 Any set X equipped with a binary relation R induces a Galois connection $\lambda \dashv \rho$ (with λ the left and ρ the right Galois maps) on subsets $U, V \subseteq X$ by $\lambda(U) = \{x \in X \mid \forall u \in U, uRx\}$ and $\rho(V) = \{x \in X \mid \forall v \in V, xRv\}$. Moreover, the compositions $\lambda\rho$ and $\rho\lambda$ are both closure operators.

These maps behave similarly to the maps l and r that we encountered in the Urquhart section. As subset $U \subseteq X$ is called stable if $U = \lambda\rho U$. Just as then, we have the following result.

Lemma 4.18 Let X be a set with a binary relation R defined on it, and let $\lambda \dashv \rho$ be the induced Galois connection. Then the stable subsets of X form a complete lattice under inclusion where meets are intersections and $A \vee B := \lambda\rho(A \cup B)$, for any stable sets A and B .

The operator $\lambda\rho$ does not, in general, distribute over unions. For instance, consider the 4-element ‘diamond shaped’ lattice $L = \{0, a, b, 1\}$ where $\lambda\rho$ is generated with the lattice ordering. Then by a trivial calculation $\lambda\rho\{a, b\} = L$ whereas $\lambda\rho\{a\} \cup \lambda\rho\{b\} = \{a, b, 1\}$. So, continuing the pattern established for boolean algebras and distributive lattices, we consider certain subsets of the closure system induced by the $\lambda\rho$ operator. Indeed, we shall consider the collection of stable compact-open subsets of an ordered topological space.

4.2.2 Hartonas Spaces

As always, we construct lattices given certain topological spaces and we construct topological spaces given certain lattices. We first describe how to go from lattices to topological spaces. We have already stated that the underlying set for the dual space is to be the set of all filters of L . The space also then comes equipped with the inclusion order. It remains to describe the topology, but we put off this issue until later. The reason for this is that we will have to make a choice about those subsets of our space which are going to be chosen to obtain its dual lattice, and these sets need to interact with the topology, the order and the associated closure operator (induced from the order). Essentially, at this point, we don’t know which sets to choose as the open sets. We employed similar tactics in previous sections, where we tried to understand the representation first, and then obtained a topology by making a clever choice for a basis (or a subbasis). Now, in keeping with the idea outlined, we are not going to try and represent that whole lattice all at once. Instead we are going to represent the meets and the joins separately. Let X_L be the set of all filters of a bounded lattice L . As we described before, we may then obtain a Galois connection $\lambda \dashv \rho$ from the inclusion order and these maps in turn induced the closure operator $\lambda\rho$. Recall that, in all of the previous representations we have encountered, from boolean algebras to bounded lattices, the representation map has remained essentially unchanged: A lattice element is represented by the collection of elements in the dual space which contain that lattice element. The only situation where we needed to be a little careful was in Urquhart’s case, because the elements in his dual space had two parts. We

would be remiss if we did not at least try to make a similar approach work here. We thus tentatively define a map $h : L \longrightarrow \wp(X_L)$ by

$$h(a) := \{x \in X_L \mid a \in x\} = X_a.$$

The hope is then that the collection $\{X_a\}_{a \in L}$, henceforth denoted by L_X , is indeed a meet-semilattice. The following lemma shows that this is so. Note that we have broken our notational convention to a certain degree by using L_X instead of L_{X_L} . The reason for this is that we do not want to be too clumsy - this will become clearer after the next two lemmas. This break of convention is unique to this section.

Lemma 4.19 (L_X, \cap) is a meet-semilattice.

Proof. We need merely demonstrate closure under intersection. Let $a, b \in L$. Then

$$\begin{aligned} X_a \cap X_b &= \{x \in X_L \mid a \in x\} \cap \{x \in X_L \mid b \in x\} \\ &= \{x \in X_L \mid (a \in x) \text{ and } (b \in x)\} \\ &= \{x \in X_L \mid a \wedge b \in x\} \\ &= X_{a \wedge b} \end{aligned}$$

since x is a filter. Note finally that this semilattice trivially has the unit $X_1 = X_L$. \square

Thus we have found a good candidate to represent the meets of L . Hartonas now looks for another semilattice, one which might represent the joins of L . But how to find such a semilattice? A clue appears when one observes that $X_a = \{x \in X \mid \uparrow a \subseteq x\}$. This makes it somewhat clearer as to what we should choose for the remaining semilattice, if we keep in mind that they should be order-inverses of each other. Define $X^a := \{x \in X \mid x \subseteq \uparrow a\}$. If we chose this intelligently, then the collection of the X^a 's should also be a meet-semilattice under intersection, now representing the joins in L . Denote the collection $\{X^a\}_{a \in L}$ by L^X .

Lemma 4.20 (L^X, \cap) is a meet-semilattice.

Proof. As before, it suffices to show closure under intersection. Indeed,

$$\begin{aligned} X^a \cap X^b &= \{x \in X \mid x \subseteq \uparrow a\} \cap \{x \in X \mid x \subseteq \uparrow b\} \\ &= \{x \in X \mid x \subseteq \uparrow a \cap \uparrow b\} \\ &= \{x \in X \mid x \subseteq \uparrow (a \vee b)\} \\ &= X^{a \vee b}. \end{aligned}$$

Again, $X^0 = X_L$ is trivially the unit in this semilattice. \square

So far so good; we have split up our lattice L into two meet-semilattices as desired. So the next question is: How can we put it back together? That is, how can we relate the joins and the meets? One way to do so is to follow Goldblatt [Gol75] and introduce a special operator, called an orthonegation operator, which we define next.

Definition 4.21 A unary operator on L , denoted by $'$ is called an orthonegation operator if it satisfies the following, for any $a, b \in L$.

1. $a'' = a$.
2. $(a \vee b)' = a' \wedge b'$.
3. $(a \wedge b)' = a' \vee b'$.

This links the joins and the meets together in an obvious way. Indeed, Goldblatt was able to avoid representing the joins by using such an operator. From the above construction of the two semilattices it is clear that Hartonas is not following this route since we have already gone to the trouble of constructing a representation for the joins. We simply need a way to connect L_X and L^X . This is where the Galois connection comes in. Each of the semilattices L_X and L^X are of course equipped with semilattice-orderings (which is just inclusion) that they inherit from the inclusion ordering on the set of filters of L . What we shall show is that there is a Galois connection between these orders. Indeed, we shall show that this is stronger than an ordinary Galois connection; we shall show that the Galois maps λ and ρ are inverses of each other in this case (this is sometimes called a duality for the partial orders). But before we do that we prove a lemma about Galois connections that are dualities in general - that is, Galois connections for which the Galois maps are inverses of each other. Note that we are abusing the usage of the word duality here. The usage of the word duality here is only a slight abuse because of the link that Galois connections have with adjoint functors.

Lemma 4.22 Let K and S be two meet semilattices with a Galois connection $\lambda \dashv \rho$ from S to K , i.e. $S \xrightarrow{\lambda} K^{op}$ and $K^{op} \xrightarrow{\rho} S$ such that $\lambda = \rho^{-1}$. Then each of S and K is a full lattice, where joins in K are defined, for any $a, b \in K$, by $a \vee b := \lambda(\rho a \wedge \rho b)$.

Proof. From $\rho a \wedge \rho b \leq \rho a$, it follows that $a = \lambda \rho a \leq \lambda(\rho a \wedge \rho b)$, so that $\lambda(\rho a \wedge \rho b)$ is an upper bound of a, b . If $a, b \leq m$, then $\rho m \leq \rho a \wedge \rho b$, hence $\lambda(\rho a \wedge \rho b) \leq \lambda \rho m = m$, so that $\lambda(\rho a \wedge \rho b)$ is the least upper bound of a and b . A dual argument shows that S is a full lattice. \square

Before we can show that the Galois connection between the semilattices L_X and L^X is a duality, we need to show that there is indeed a Galois connection to begin with. Fortunately for us this is automatic since a Galois connection for the semilattice orderings on L_X and L^X comes from the Galois connection $\lambda \dashv \rho$ induced by the inclusion ordering on X_L . The next lemma shows that this Galois connection is a duality.

Lemma 4.23 Let L be a bounded lattice, and let L_X and L^X be constructed as above. Then the Galois connection $\lambda \dashv \rho$ is a duality, i.e. λ and ρ are inverses of each other.

Proof. We need to establish that $\rho X_a = X^a$ and that $\lambda X^a = X_a$. For the first identity, if $x \in \rho X_a$, this means that x is a lower bound of X_a , which is the upper closure of the principal filter $\uparrow a$, and therefore $x \leq \uparrow a$ showing that

$x \in X^a$. Conversely, any filter contained in $\uparrow a$ is below any filter in X_a , hence it is a lower bound of X_a . The proof of the other identity is done dually. \square

So we have shown that the Galois connection between X_L and X^L is a duality. This means that each of X_L and X^L is a full lattice. Moreover, by applying λ to both sides of $\rho X_a = X^a$ and using the above lemma, we see that $\lambda \rho X_a = X_a$, which shows that the sets X_a are all stable under the closure operator $\lambda \rho$. Further application of the above lemma gives $X_a \vee X_b = \lambda(\rho X_a \cap \rho X_b)$. We have finally come full circle. The next lemma shows that the full lattice L_X is precisely what we want.

Lemma 4.24 Let L be a bounded lattice, and let X_L be the set of filters of L . Then the representation map $h : L \longrightarrow L_X$ is a lattice isomorphism.

Proof. Firstly, h is injective. Indeed, $X_a = X_b \Rightarrow \uparrow a \subseteq \uparrow b \subseteq x$ for all $x \in X$. In particular, $\uparrow a \subseteq \uparrow b$, thus $b \in \uparrow a \Rightarrow a \leq b$. Similarly, $b \leq a$, showing that $a = b$. Also, $X_a \vee X_b = \lambda(\rho X_a \cap \rho X_b) = \lambda(X^a \cap X^b) = \lambda X^{a \vee b} = X_{a \vee b}$ showing that h is a lattice homomorphism. Surjectivity is tautological from the definition of L_X . Thus h is a lattice isomorphism. \square

We discovered in Lemma 4.18 that the stable sets of a closure operator form a complete lattice under inclusion. Further, since all the sets X_a are stable, there is a natural embedding of L_X in this complete lattice. We have obtained a way to represent our lattice L , but so far we have ignored the topology on X_L . We now remedy the problem by letting the family $\{X_a\}_{a \in L} \cup \{-X_b\}_{b \in L}$ be a subbasis for a topology on X_L . This is now not an odd choice given our understanding of the representation of L . It is also analogous to our choice in the distributive case and in the boolean case, the only difference being in the closure operator used. The next lemma shows explicitly the connection between our choice for the topology and the lattice structure. As with all the proofs in this section, it is taken directly from Hartonas' paper.

Lemma 4.25 The stable compact-open subsets of X_L are exactly the sets X_a , $a \in L$.

Proof. Every X_a is trivially clopen by way of the definition of our topology. This means each X_a is compact, since it is a closed subset of a compact Hausdorff space, where the Hausdorff property is a consequence of the fact that a Stone space is totally disconnected.

For the converse, let $U \subseteq X_L$ be a stable compact-open set. Let $x = \bigcap U = \bigcap \{z \mid z \in U\}$. The intersection is never empty since for any $z \in U$, $1 \in z$. Then $y \leq U$ iff $y \leq x$ since x is the greatest lower bound. Then $\rho U = \{x \in X \mid \forall u \in U, x \leq u\}$. So the elements of ρU are precisely those elements below all elements of U , or equivalently, those elements below x . Thus $\rho U = \downarrow x$. Then $\lambda \downarrow x = \{y \in X \mid \forall v \in \downarrow x, v \leq y\}$. So precisely the elements above all of $\downarrow x$ are present here, or equivalently, elements above x . Thus, $\lambda \rho U = U = \uparrow x$.

We still need to show that U may be written in the form X_a , for some $a \in L$. Firstly, we notice that $U = \bigcap_{a \in x} X_a$, where $x = \bigcap U$. Indeed, if $u \in U$,

then $x \subseteq u$ and hence for any $a \in x$, $u \in X_a$. Since this holds for any a , $u \in \bigcap_{a \in x} X_a$. Conversely, if $u \in \bigcap_{a \in x} X_a$, then u is a filter that contains every element of x , or equivalently, $x \subseteq u$. But $U = \uparrow x$ so that $u \in U$.

By De Morgan's laws, it follows that $-U = \bigcup_{a \in x} (-X_a)$. Since U is clopen, so is $-U$ and thus $-U$ is compact. But then there is a finite subcover of $\{-X_a\}_{a \in x}$ which covers $-U$. Let these sets be called X_{a_1}, \dots, X_{a_n} . Then $U = \bigcap_{i=1}^n X_{a_i}$. Letting $a = a_1 \wedge \dots \wedge a_n$, it follows that $U = X_a$, completing the proof. \square

The concrete case seems easy enough. We now wish to extract those properties we need on an abstract topological space so that we may 'go back'. We state them in the following definition (for condition 5 below we recall Definition 1.16).

Definition 4.26 Let (X, Ω, \leq) be an ordered topological space, with $\lambda\rho$ the closure operator associated to \leq . Denote the family of stable compact-opens of X by \mathfrak{C} . Then X is called a Hartonas space if it satisfies the following:

1. X is compact and totally disconnected.
2. The family $\{A_i\}_{i \in I} \cup \{-A_j\}_{j \in J}$ forms a subbasis for the topology, where $A_i, A_j \in \mathfrak{C}$ for all i, j .
3. If $A, B \in \mathfrak{C}$ then $\lambda\rho(A \cup B) \in \mathfrak{C}$.
4. If A is stable, then $A = \lambda\rho(x)$, for some $x \in X$.
5. The family \mathfrak{C} is compact-dense in the lattice of stable subsets of X .

Lemma 4.27 If L is a bounded lattice, then X_L is a Hartonas space.

Proof. Goldblatt proves in [Gol75] that X_L is compact and totally disconnected. The stable compact-opens forms a basis by definition of the topology on X_L and Lemma 4.25. Next, observe that in general, if A is stable, then $A = \lambda\rho U$ for at least one $U \subseteq X_L$ (in particular for A itself). Consider therefore $\lambda\rho A$ for some $A \subseteq X_L$. Note firstly that the intersection of all the elements of A (all of them filters) is non-empty because of the top element, and it is trivially also a filter. Call this filter x . Secondly, ρA is the collection of filters contained in every member of A , or equivalently, all those filters contained in x . Thus $\rho\{x\} = \rho A$ which implies $\lambda\rho\{x\} = \lambda\rho A$, and by definition, x is some element in X_L . To show condition 5, let A be stable. Then $A = \lambda\rho\{x\}$ for some $x \in X_L$ as indicated. Let $B := \{b \in L \mid b \in x\}$. Then trivially $\lambda\rho\{x\} = \bigwedge_{b \in B} X_b$. We prove the last part in the contra-positive manner. Let $\{\lambda\rho\{x_i\}\}_{i \in I}$ be a collection of stable sets with the property that for every finite $F \subseteq I$, there is a $y \in \bigwedge_{i \in F} \lambda\rho\{x_i\}$ with $a \notin y$. In particular then, for each $\lambda\rho\{x_i\}$ individually, there is a $y \in \lambda\rho\{x_i\}$ with $a \notin y$. But since $x_i \subseteq y$, $a \notin x_i$ for each i . Let $\lambda\rho\{x\} = \bigwedge_{i \in I} \lambda\rho\{x_i\}$, for some $x \in X_L$. Then we have $a \notin x$. Thus $x \notin X_a$ and hence $\bigwedge_{i \in I} \lambda\rho\{x_i\} \not\subseteq X_a$. \square

Lemma 4.28 Let (X, Ω, \leq) be a Hartonas space. Then the collection of stable compact-opens forms a bounded lattice.

Proof. Lemma 4.18 already guarantees that the stable sets form a lattice. Next, we note that \emptyset and X itself are stable and compact-open. Further, the intersection (which is the meet operation here) of two compact-open sets is trivially compact-open. It remains to show that if A and B are stable compact-open sets, then $\lambda\rho(A \cup B)$ will also be compact-open. But this is so by definition of a Hartonas space. \square

4.2.3 Representation Results

Theorem 4.29 Let L be a bounded lattice. Then the map

$$h : L \rightarrow L_X,$$

given by $a \mapsto X_a$, is an isomorphism of L onto the lattice of stable compact-open subsets of its dual space X_L .

Proof. The proof follows immediately from Lemma 4.24 and Lemma 4.25. \square

The proof of the next representation theorem is much more involved, and for a proof we refer the reader to Lemma 4.5 of [Har97].

Theorem 4.30 Let (X, Ω, \leq) be a Hartonas space. Then the map

$$f : X \rightarrow X_L,$$

given by $x \mapsto F_x$, where F_x is the collection of all stable compact-opens in X that contain x , is an order-homeomorphism of X onto the dual space of the lattice of stable compact-opens of X .

The most telling difference between Hartonas' approach and the strategy employed by Stone, Priestley and Urquhart is that Hartonas never uses the axiom of choice during the representation. The obvious downside then is that Stone's and Priestley's results cannot be seen as special cases of Hartonas' result. The main idea that Hartonas wanted to convey was that every lattice is a sublattice of a lattice induced by a closure operator (the $\lambda\rho$ operator in this case, and the topological closure in the boolean case for instance). As a final note we mention that the construction is functorial, but we omit the proof of this fact as the emphasis of this thesis is to study the representation itself and the various important operators that lattices are normally endowed with.

4.3 Bounded Lattice Spaces

4.3.1 Introduction

In this section we discuss the work done by Moshier and Jipsen in [Mos09]. We take a second to note that the paper on which this section is based is still in preprint. Their work represents a departure from the philosophy of Hillary Priestley [Pri70] in the sense that they sought to find a ‘pure’ correspondence between lattices and certain topological spaces. In their own words: ‘Is there a subcategory of **Top** that is dually equivalent to **Lat**?’. Of course **Top** means the category of topological spaces with continuous maps and **Lat** is the category of bounded lattices with lattice homomorphisms that preserve the bounds.

Thus far, we have been ‘cheating’ a bit in our attempts to solve this problem. We either considered only a special class of lattices (such as Stone’s work on boolean algebras) or we generalized the concept of a topological space (by endowing it with a partial order). All of the different approaches taken were studied to varying levels of detail in the preceding sections. So how do Moshier and Jipsen go about obtaining their duality? Again we quote directly from their paper: ‘We take a different path via purely topological considerations that simplifies Hatonas’ duality by eliminating the need for an auxiliary binary relation ... This establishes an affirmative answer to our original question with no riders: the dual category to **Lat** is a subcategory of **Top** *simpliciter*’. In fact, Moshier and Jipsen did a little more than this. In the process of discovering the aforementioned, they also arrive at a duality for semilattices with unit. Indeed, they make use of this duality to obtain their results on bounded lattices. For this reason, we shall discuss their work done on semilattices first, and then see how it may be specialized to the bounded lattice case.

Even though the results obtained by Moshier and Jipsen are functorial, we shall only study the representation of the lattices themselves. Throughout, only bounded lattices and semilattices with a unit are considered. The proofs and definitions in this section come directly from [Mos09], where some have been expanded upon for clarification.

4.3.2 Semilattices and SL Spaces

Let us then begin with a meet semilattice with unit M , where we choose the letter M to distinguish it from the case of a full lattice. In the introduction to this section we mentioned that we are going to follow a similar idea as Hartonas, in the end simplifying his work. We do not lose therefore all the intuition that we have gained there, and therefore define X_M to be the set of all filters in M . As usual this is to be the underlying set of the dual space. We have but to decide how to topologize X_M . Moshier and Jipsen follow an interesting route here. We first make use of the following known lemma, a proof of which may be found in [Dav90].

Lemma 4.31 If M is a meet semilattice (with unit), then the collection of filters in M , ordered by inclusion, is a dcpo.

We refer the reader to Definition 1.25 for the definition of a dcpo. The above lemma implies that we may imbue X_M with a special topology, called the Scott topology.

Definition 4.32 Let L be a dcpo. Then there is a topology on L , called the Scott topology, where $U \subseteq L$ is defined to be open if it is an up-set and has non-empty intersection with any directed set D whose supremum is in U .

The question is, which properties does our newly created space possess? Recalling the introduction to this section, we want to try and generalize the ideas of Stone's representation of distributive lattices. We refer the reader to [Sto36] for a proof of Stone's duality for bounded distributive lattices and merely state it here.

Theorem 4.33 Every bounded distributive lattice is isomorphic to the lattice of compact opens of its dual space. Conversely, every spectral space is homeomorphic to the dual space of its lattice of compact opens. Stated in its full categorical version: The category **DL** of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category **Spec** of spectral spaces and spectral maps.

For readers who are not that familiar with Stone's work, we provide the following definition.

Definition 4.34 A topological space is called a spectral space if it is sober (Definition A.15) and if the compact open sets form a basis that is closed under finite intersection.

Our attempt then is to try and find a link between X_M and these spectral spaces. The first thing that stands in our way is the fact that spectral spaces are supposed to accompany distributive lattices, whereas we are working with meet semilattices. This motivates the next discussion.

For a meet semilattice M , let $DL(M)$ denote the free distributive lattice over M . That is, $DL(M)$ is concretely built as the collection of finite unions of principal down-sets in M . Join is union and meet is computed in general by extension of $\downarrow a \cap \downarrow b = \downarrow (a \wedge b)$. We omit the proof of the fact that this indeed makes $DL(M)$ into a distributive lattice. The map $a \mapsto \downarrow a$ is the semilattice embedding $M \rightarrow DL(M)$.

So now we have a distributive lattice based on our meet semilattice, and we may use Stone's result to associate a spectral space with this lattice. To borrow the notation used by Moshier and Jipsen, call this space $\text{spec}(DL(M))$. We now set out to discover if there is a link between X_M and $\text{spec}(DL(M))$. To do so, we need the assistance of the following lemma.

Lemma 4.35 Let P be a partially ordered set. Then the opens of X_P , under the Scott topology, are order-isomorphic with the collection of down-sets of P .

Proof. Suppose $D \subseteq P$ is a lower set. Define $\mathcal{U}_D := \{F \in X_P \mid D \cap F \neq \emptyset\}$. Clearly, \mathcal{U}_D is an upper set of filters. Moreover, if D is a directed set of filters,

then $\bigcup \mathfrak{D} \in \mathcal{U}_D$ then for some $F \in \mathfrak{D}$, $F \in \mathcal{U}_D$. So \mathcal{U}_D is Scott open. Suppose that \mathcal{U} is a Scott open set of filters. Define $D_{\mathcal{U}} := \{a \in P \mid \uparrow a \in \mathcal{U}\}$. Since \mathcal{U} is an upper set, this is a lower set. Because any filter F is the directed union of principal filters contained in it, $F \in \mathcal{U}$ iff there exists $a \in F$ such that $a \in D_{\mathcal{U}}$. Likewise, for a lower set D , $a \in D$ iff $\uparrow a \in \mathcal{U}_D$. So the constructions $D_{\mathcal{U}}$ and \mathcal{U}_D are order preserving bijections. \square

We prove the desired lemma.

Lemma 4.36 For a meet semilattice M , it follows that X_M is homeomorphic to $\text{spec}(DL(M))$.

Proof. A filter F in M determines a filter basis $\{\downarrow a \mid a \in F\}$ in $DL(M)$, which evidently generates a prime filter. A prime filter $P \subseteq DL(M)$ determines a filter $\{a \in M \mid \downarrow a \in P\}$ in M . These are exactly checked to be inverses of each other. It is also routine, using Lemma 4.35, to check that these two maps are continuous. \square

So there is a definite connection here to Stone's result. The dual spaces that we are creating here are spectral spaces! That means that they inherit all the properties that spectral spaces possess. Next we state a few definitions that we shall require.

Definition 4.37 A subset F of a topological space X is called saturated if it is an up-set with respect to the specialization ordering on X .

Definition 4.38 Let X be a topological space. $F \subseteq X$ is called a filter in X if F is saturated and if every two elements in F have a lower bound in F .

Note that the above definition is somewhat different from the definition of a filter we have been using so far. This is forced upon us here because the specialization ordering is not a lattice ordering in general. We shall be interested in the following subsets of X_M .

- $K(X_M)$: The collection of compact saturated subsets of X_M .
- $O(X_M)$: The collection of open subsets of X_M .
- $F(X_M)$: The collection of filters in X_M .

We shall denote intersections of the above collections by concatenation. For instance, $OF(X) = O(X) \cap F(X)$. It might seem like we are 'cheating' by caring about the specialization order, given that we wish to construct a 'pure' correspondence between lattices and topological spaces. The key difference is that the specialization order is an inherent property of a topological space. Moreover, the specialization order is a pre-order, not a partial order. The next few lemmas illustrate some properties of the above three collections.

Lemma 4.39 Any open set in a topological space is saturated.

Proof. Let U be an open set in a topological space X , and let $x \in U$. We need to show that $x \leq y$ implies that $y \in U$ for any y . Note that the proof follows trivially if $U = X$ or if the specialization order is discrete. Thus let $x \leq y$. Then by definition, every open set containing x also contains y . In particular then, $y \in U$. \square

It is also trivial to see that any intersection of saturated sets is again saturated.

Lemma 4.40 In a topological space X , let F_1, \dots, F_m be pairwise incomparable filters (with respect to specialization). Then $F_1 \cup \dots \cup F_m$ is compact if and only if each F_i is a principal filter.

Proof. Firstly, each principal filter $\uparrow x$ is compact. Indeed, any collection of open sets covering $\uparrow x$ must necessarily contain a member, say U , that contains x . But, since U is open, it is saturated, and hence is an upper set with respect to specialization. Hence $\uparrow x \subseteq U$.

Conversely, let L be the collection of open sets U such that $F_m \setminus U \neq \emptyset$. For $x \in F_m$, there is an element $y \in F_m$ so that $x \not\leq y$. Thus there is a U for which $x \in U$ but $y \notin U$. For $x \in F_i$ ($i < m$) the filters are pairwise incomparable, so there is an element $y \in F_m$ so that $x \not\leq y$. Again, there is an open U separating x from y . Thus L is an open cover of $F_1 \cup \dots \cup F_m$. Suppose $U, V \in L$. Then there are elements $x \in F_m \setminus U$ and $y \in F_m \setminus V$. Because F_m is a filter, there is also an element $z \in F_m$ below both x and y . Hence $z \in F_m \setminus (U \cup V)$. So L is directed. By construction, no $U \in L$ covers $F_1 \cup \dots \cup F_m$. \square

Definition 4.41 A point x in a topological space X is called finite if $\uparrow x$ is open in X . The collection of all finite points of X is denoted $\text{Fin}(X)$.

Using the above lemma, we see that the compact filters are all principal. We are also able to prove the following corollary.

Corollary 4.42 Let X be a topological space. Then $KOF(X)$ is in an order-reversing bijection with $\text{Fin}(X)$.

Proof. Let $F \in KOF(X)$. Then since F is a compact filter, it is principal. That is, $F = \uparrow x$ for some $x \in X$. But F is also open, hence x is finite. Conversely, if x is finite, then by definition $\uparrow x$, which is trivially a filter, is open. By Lemma 4.40, $\uparrow x$ is compact, and hence $\uparrow x \in KOF(X)$. Finally $x \leq y$ iff every open set containing x also contains y . In particular then, $y \in \uparrow x$ and hence $\uparrow y \subseteq \uparrow x$, reversing the order. \square

Theorem 4.43 For a topological space X , the following are equivalent:

1. X is spectral and $OF(X)$ forms a basis that is closed under finite intersection;
2. X is spectral, $OF(X)$ forms a basis, X is a meet semilattice with respect to specialization and X has a least element;
3. X is sober and $KOF(X)$ forms a basis that is closed under finite intersection.

Proof. Suppose (3) holds. Then the compact opens and the open filters separately form bases. Furthermore, if K and H are compact opens, then $K = F_1 \cup \dots \cup F_m$ for some compact open filters F_i and likewise $H = G_1 \cup \dots \cup G_n$. Since each set $F_i \cap G_j$ is compact, so is $K \cap H$. Similarly, X is the empty intersection of open filters, and hence is also a compact open filter. So X is spectral and has a least element. Since $KOF(X)$ is closed under finite intersection, $Fin(X)$ is itself a directed subset of X . By sobriety the supremum exists, which must be the greatest element of X . For $x_0, x_1 \in X$, consider $B_{x_0, x_1} = \{a \in Fin(X) \mid x_0, x_1 \in \uparrow a\}$. Because of (3), this is a directed set which has a supremum $y = \bigcup^\uparrow B_{x_0, x_1}$. If $y \in U$, then $y \in \uparrow a \subseteq U$ for some $a \in B_{x_0, x_1}$. So $y \leq x_0$ and $y \leq x_1$. Now consider y' , a lower bound of x_0 and x_1 . Then $y' \in U$ implies that $y' \in \uparrow a \subseteq U$ for some finite a . But $a \in B_{x_0, x_1}$, so $y' \leq y$.

Suppose (2) holds. The least element of X ensures that X itself is a filter. Suppose F and G are both open filters. Then $F \cap G$ is open and is a filter because X is a meet semilattice.

Suppose (1) holds. Spectral spaces are sober. Any compact open K equals $F_1 \cup \dots \cup F_m$ for some open filters F_i . These can be chosen to be pairwise incomparable. So $KOF(X)$ forms a basis. Evidently, a finite intersection of open filters is an open filter. Hence (3) holds. \square

Definition 4.44 A topological space X is called an *SL* space if it is sober and $KOF(X)$ forms a basis that is closed under finite intersection.

We shall eventually show that X_M is an *SL* space. In all our previous chapters and sections, the topology on the dual space was also intimately linked to the representation map for the lattice. The reader might wonder if this is the case here as well. Indeed, the collection

$$X_a := \{F \in X_M \mid a \in F\}$$

where a ranges over M is a basis for the topology on X_M . We omit the proof of this fact.

Our next mission is to show that the dual space X_M of a meet semilattice M is an *SL* space. Since X_M is spectral, using Lemma 4.43, we see that it remains only to show that $KOF(X_M)$ is a basis for the topology, that X_M is a meet semilattice with respect to specialization and that it has a least element.

Lemma 4.45 If M is a meet semilattice then X_M is a meet semilattice with respect to specialization.

Proof. We need merely that any two elements in X_M has an infimum. So let $x, y \in X_M$. These are filters in M , and so is $z := x \cap y$. We show that z is the infimum of x and y . Firstly $z \leq x$, for if it were not so, there would be a basic open set X_a with $a \in M$, such that $z \in X_a$ but $x \notin X_a$. Thus $a \in z$ but $a \notin x$, which is impossible. Similarly, $z \leq y$. Let w be any other lower bound of x and y . We want $w \leq z$. Suppose it were not so, so that there is a basic open set X_a , with $a \in M$, such that $w \in X_a$ but $z \notin X_a$. Thus $a \in w$ but $a \notin z$. Now since w is a lower bound for both x and y , $a \in x$ and $a \in y$ or equivalently,

$a \in x \cap y = z$, which is a contradiction. \square

Lemma 4.46 For a meet semilattice M , X_M is an SL space.

Proof. Since $\{1\}$ and M are the least- and greatest elements respectively, it remains to show that the open filters form a basis. But the sets X_a are a basis, and these clearly are filters. \square

So we have established that the dual spaces of meet semilattices are precisely the SL spaces. We now want to know how to create a dual lattice L_X given an SL space X . We are going to look to the collection $KOF(X)$ to assist us.

Lemma 4.47 If X is an SL space, then $KOF(X)$ is a meet semilattice.

Proof. The meet operation is given by intersection, and by definition of X , the family $KOF(X)$ is closed under finite intersection. X is trivially the unit in $KOF(X)$. \square

Let us next examine the concrete case.

Lemma 4.48 If M is a meet semilattice and X_M is its dual space, then the families $KOF(X_M)$ and X_a , where a ranges over M , coincide.

Proof. Trivially, X_a is an open filter for each $a \in M$. To prove that it is also compact, let \mathcal{U} be an open covering of X_a , for some arbitrary but fixed a . Then, in particular, $\uparrow a \in U$ for some element $U \in \mathcal{U}$. Since each open set is saturated, all filters containing $\uparrow a$ must also be in the open set U . But then the single set U is a finite subcovering of X_a . Conversely, if $V \in KOF(X_M)$, then the fact that V must be a principal filter ensures that $V = X_a$ for some $a \in M$. \square

If X is an SL space, its dual lattice, denoted M_X is defined as the family $KOF(X)$. We finally reach the representation theorems.

Theorem 4.49 If M is a meet semilattice, then there is a semilattice isomorphism of M onto M_{X_M} given by

$$a \mapsto X_a$$

where $X_a := \{F \in X_M \mid a \in F\}$.

Proof. We have already seen that the mapping is a homomorphism, and it is clearly onto. Further, since $X_1 = M$, the top element is preserved. It remains to show injectivity. Now if $X_a = X_b$ then, in particular, $\uparrow a \in X_b$ so that $a \leq b$ and similarly $\uparrow b \in X_a$ so that $b \leq a$. Thus $a = b$. \square

Theorem 4.50 If X is an SL space then there is a homeomorphism of X onto X_{M_X} given by

$$x \mapsto F_x$$

where $F_x := \{F \in KOF(X) \mid x \in F\}$.

Proof. Firstly, notice that F_x is indeed trivially a filter in M_X . To show injectivity, suppose that $F_x = F_y$. Then $\uparrow x$ is a member of $KOF(X)$ by Lemma 4.40. Thus $x \leq y$. Similarly, $y \leq x$ so that $x = y$, because X is a meet semi-lattice with respect to specialization. To show continuity, let $x \in X$ and let U be any basic open set containing F_x . Then $U = X_A := \{F \in X_{M_X} \mid A \in F\}$ for some $A \in M_X$. We show that A is the desired open set mapping into X_A . Indeed, if $y \in A$, then $A \in F_y$ and thus $F_y \in X_A$, so that our map is continuous at x . Further, the mapping is also open. It suffices by injectivity to show that basic open sets are mapped to basic open sets. We shall show that $A \in KOF(X)$ gets mapped precisely to X_A . Indeed, if $x \in A$, by the same argument as above, $F_x \in X_A$. Conversely, if $x \notin A$, then $A \notin F_x$ and hence $F_x \notin X_A$. Finally we show that it is onto. Indeed, if $Y \in X_{M_X}$, then by definition Y is a filter in M_X . We may write $Y := \bigcup_{A \in Y} \{\uparrow A\}$. But by Corollary 4.42 there is a finite point $x \in X$ associated with each $A \in Y$. Let the collection of these finite points be $\{x_i\}_{i \in I}$. By Lemma 4.52 X is a complete lattice with respect to specialization. Then trivially $\bigwedge_{i \in I} x_i \mapsto Y$. \square

4.3.3 Bounded Lattices and BL Spaces

Now we specialize the above results so that similar ideas apply to bounded lattices. The first thing we do is expand on the notion of saturation. Consider the following definition.

Definition 4.51 A subset of a SL space is called F -saturated if it is an intersection of open filters.

Lemma 4.52 Let X be an SL space and let $U \subseteq X$ be F -saturated. Then U is a filter.

Proof. Since an SL space has a top element, the intersection of any collection of filters is never empty. U is an up-set since it is the intersection of a collection of up-sets. Finally, it is closed under finite meets, since any two elements in U lies in one of the filters V whose intersection yields U , and since V is a filter, it contains the meet of those two elements. Hence U is a filter. \square

Let $FSat(X)$ denote the collection of all F -saturated subsets of an SL space X . Define

$$fsat(A) := \bigcap \{F \in OF(X) \mid A \subseteq F\}.$$

One may then prove the following, which we state here without proof.

Lemma 4.53 If X is an SL space, then $FSat(X)$ forms a complete lattice, with meet given by intersection and joins defined by $\bigvee \{A_i\} := fsat(\bigcup A_i)$, for $A_i \in FSat(X)$ for each i .

In short, the above tells us that $fsat$ is a closure operator; in any space with a greatest element, $fsat$ produces a filter.

Lemma 4.54 If X is an SL space, then X is a complete lattice with respect to specialization. Moreover, for a compact set A , $fsat(A)$ is compact, hence is a principal filter, and $\min fsat(A) = \bigwedge A$.

Proof. The earlier proof that X is a meet semilattice generalizes to arbitrary meets. That is, for $A \subseteq X$, let $B_A^* := \{F \in KOF(X) \mid A \subseteq F\}$, writing B_x^* for singletons. Each $F \in B_A^*$ is principal, so $B_A := \{\min F \mid F \in B_A^*\}$ is directed. Hence $x := \bigvee^\uparrow B_A$ exists. Obviously, x is a lower bound of A . If x' is another lower bound, then $B_{x'} \subseteq B_A$. So $\bigvee^\uparrow B_{x'} \leq x$. But since $KOF(X)$ is a basis for the topology, $X' \leq \bigvee^\uparrow B_{x'}$.

If A is compact and $A \subseteq F$ for an open filter F , then by compactness there is some $G \in KOF(X)$ for which $A \subseteq G \subseteq F$. Thus $fsat(A) = \bigcap B_A^* = \uparrow \bigwedge A$. \square

$FSat(X)$ has a bit more concrete structure. In particular, suppose \mathfrak{D} is a directed set of open filters. Then the union is also an open filter. Hence this union is F -saturated. In other words, in $FSat(X)$ a directed join of open filters is simply a union.

We now consider what conditions on an SL space are necessary and sufficient for $KOF(X)$ to form a lattice, not just a semilattice.

Theorem 4.55 For an SL space, the following are equivalent.

1. $KOF(X)$ forms a sublattice of $FSat(X)$.
2. $KOF(X)$ forms a sublattice of $FSat(X)$.
3. $fsat(U)$ is again open for any open U .

Proof. Suppose (1) holds. For compact open filters F and G , the join in $FSat(X)$ is $fsat(F \cup G)$. But $F \cup G$ is compact, hence by Lemma 4.54 so is $fsat(F \cup G)$. Likewise, $fsat(\emptyset)$ is the least element of $FSat(X)$ and is compact.

Suppose (2) holds. Consider an open set U . Since X is a complete meet semilattice, U generates a filter F . That is, $x \in F$ iff for some $y_0, \dots, y_m \in U$ it follows that $y_0 \wedge \dots \wedge y_m \in x$. Evidently, it suffices to show that F is open, for then $F = fsat(U)$. For $x \in F$, pick $y_0, \dots, y_m \in U$ that meet below it. According to Lemma 4.54, $y_i = \bigvee^\uparrow B_{y_i}$. But U is open. So we may choose an element $a_i \in B_{y_i} \cap U$ in place of y_i . Now, $\uparrow a_i$ is a compact open filter, so (2) tells us that $fsat(\uparrow a_0 \cup \dots \cup \uparrow a_m) = \uparrow (a_0 \wedge \dots \wedge a_m) \subseteq F$ is a compact open filter that contains x .

Suppose (3) holds. Then for any two open filters, $fsat(F \cup G)$ is open. It is a filter because it is non-empty. Likewise, $fsat(\emptyset)$ is open and non-empty. \square

Definition 4.56 An SL space X for which $KOF(X)$ forms a sublattice of $FSat(X)$ is called a BL space.

The next task is to show that every lattice occurs isomorphically as $KOF(X)$ for some BL space X . The basic construction is the same as in the semilattice case. Now in the case that L is a lattice, X_L has additional structure. We collect various useful facts in the following.

Lemma 4.57 Let L be a lattice. In X_L the following hold.

1. An open \mathcal{U}_D is a filter iff D is an ideal of L .

2. Finite joins of compact open filters are given by joins in L . That is, $fsat(\mathcal{U}_{\downarrow a} \cup \mathcal{U}_{\downarrow b}) = \mathcal{U}_{\downarrow (a \vee b)}$ and similarly, $fsat(\emptyset) = \mathcal{U}_0$.
3. The way below relation is given by $\mathcal{U}_D \ll \mathcal{U}_E$ iff for some finite $\{a_1, \dots, a_n\} \subseteq E$, D is a subset of $\bigcup_{i=1}^n \downarrow a_i$.

Proof. For (1), suppose D is an ideal in L , and $a \in F \cap D$ and $b \in G \cap D$. So $a \vee b \in D$, and $x \vee y \in F \cap G$. So \mathcal{U}_D is a filter of filters. Conversely, suppose that \mathcal{U}_D is a filter of filters, and $a, b \in D$. Then $\uparrow a \in \mathcal{U}_D$ and $\uparrow b \in \mathcal{U}_D$. Thus $\uparrow a \cap \uparrow b = \uparrow (a \vee b) \in \mathcal{U}_D$. That is, $a \vee b \in D$.

For (2), $\mathcal{U}_{\downarrow (a \vee b)}$ is an F -saturated set containing $\mathcal{U}_{\downarrow a} \cup \mathcal{U}_{\downarrow b}$. If \mathcal{U}_I contains $\mathcal{U}_{\downarrow a} \cup \mathcal{U}_{\downarrow b}$ then in particular, $a, b \in I$. So $a \vee b \in I$. Evidently, $\mathcal{U}_0 = \{L\}$, which is the smallest F -saturated set of filters.

The characterization of \ll in (3) is a standard fact about the Scott topology of an algebraic dcpo. \square

Lemma 4.58 For a lattice L , X_L is a BL space.

Proof. For a lattice L , it remains to check that $fsat(\mathcal{U}_D)$ is open whenever D is a lower set in L . The open filtes containing \mathcal{U}_D are bijective with the ideals containing D . So let I be the smallest ideal containing D . Then \mathcal{U}_I is evidently equal to $fsat(\mathcal{U}_D)$. \square

It is also a part of the very conditions on a BL space X that the family $KOF(X)$ is a lattice, so that we may confidently set $L_X = KOF(X)$ in this case as well. We next show the representation theorems, the proofs of which are both similar to the SL case.

Theorem 4.59 Let L be a bounded lattice. Then there is a lattice isomorphism between L and L_{X_L} given by

$$a \mapsto X_a$$

where $X_a := \{F \in X_L \mid a \in F\}$.

Theorem 4.60 Let X be a BL space. Then there is a homeomorphism between X and X_{L_X} given by

$$x \mapsto F_x$$

where $F_x := \{F \in KOF(X) \mid x \in F\}$.

Chapter 5 - Operator Duality

Now that the representations of boolean algebras, distributive lattices and bounded lattices have been obtained, we go one step further and endow the lattices under consideration with certain structure preserving maps and ask ourselves what sort of structure needs to be added onto the topology to compensate. In particular we will endow our lattices with maps which preserve the meet- and join structure, as well as the bounds. The pioneers in these ideas are Jónson and Tarski [JoT51] [JoT52], who worked on boolean algebras with operators. In particular, the notion of associating a relation to an operator is due to these authors. Further work was by Hansoul [Han83], also in the case of boolean algebras, and by Goldblatt [Gol89] in the case of bounded distributive lattices. As the reader will recall, the aim of this thesis is the application of Goldblatt's approach to the case where the lattice is an arbitrary bounded lattice. To this end, we shall find it useful to explore his ideas. This is done in the next section.

5.1 Duality for Distributive Lattices with Operators

One should keep in mind that the duality developed by Priestley is the foundation on which Goldblatt's work is built. That is, we do not alter at all the underlying duality. We merely ask ourselves which particular structure one needs to add on the topological side so that the mapping is 'preserved'. We shall discover that relations are the correct structure to endow the dual space with is a relation. Let us introduce the concepts and definitions that we shall require.

Note: All proofs and definitions in this section are due to Goldblatt, and may be found in his paper.

Definition 5.1 Let L be a bounded distributive lattice. A unary operator

$$\Diamond : L \rightarrow L$$

is a join-hemimorphism if it satisfies

1. $\Diamond 0 = 0$ and
2. $\Diamond(a \vee b) = \Diamond(a) \vee \Diamond(b)$

for any $a, b \in L$ and where 0 is the bottom. We define a meet-hemimorphism dually, that is, $\Box : L \rightarrow L$ is a meet-hemimorphism if the equations

1. $\Box 1 = 1$ and

$$2. \quad \Box(a \wedge b) = \Box(a) \wedge \Box(b)$$

hold for any $a, b \in L$, where 1 is the top.

The name ‘hemimorphism’, first coined by Halmos [Hal55], was chosen since it preserves roughly ‘half’ of the structure of a lattice. The following lemma illustrates a useful fact about hemimorphisms.

Lemma 5.2 Hemimorphisms are monotone, that is, $a \leq b$ implies $\Diamond(a) \leq \Diamond(b)$ and $\Box(a) \leq \Box(b)$.

Proof. Suppose that $a \leq b$ and that \Diamond is a join-hemimorphism. Then $b = a \vee b$. Thus $\Diamond(b) = \Diamond(a \vee b) = \Diamond(a) \vee \Diamond(b)$. Thus $\Diamond(a) \leq \Diamond(b)$.

Similarly, if \Box is a meet-hemimorphism, then $a \leq b$ implies $a = a \wedge b$. Thus $\Box(a) = \Box(a \wedge b) = \Box(a) \wedge \Box(b)$. Hence $\Box(a) \leq \Box(b)$. \square

A distributive lattice with operators is a structure (L, \Diamond, \Box) where L is a bounded distributive lattice, \Diamond is a join-hemimorphism and \Box is a meet-hemimorphism. We shall not endow a lattice with multiple join- or meet-hemimorphisms and instead restrict ourselves to a single join-hemimorphism and meet-hemimorphism. So far we have decided to associate a relation (on the topological side) with an operator (on the lattice side). But the operator has the structure preserving properties discussed above, and it is not clear at first sight which properties we would require our relation to have so that we may ‘go back’. The next few definitions (which will be motivated later) are to assist us in this matter.

Note that in the above definition, the notation xRy is to be read ‘ x is related to y , under the relation R ’. That is, $(x, y) \in R$. So we have at least pinned down which basic structure we should associate with a distributive lattice with operators: We should associate an ordered topological space with certain binary relations. Let us call a structure (X, \leq, R, Q) an ordered relational structure (we shall add the topology later).

Now the next question is: Which properties should these relations have? In particular, how should they interact with the topology on X , and how should they interact with the order? The first property that we wish the relations to have (they should have some monotonicity properties, made precise by the next definition) is not too hard to motivate if one remembers that we are going to be dealing with Priestley spaces. Recall that the clopen up-sets and their complements formed a subbasis for the topology on those spaces, and moreover, we constructed the dual lattice of X using the collection of clopen up-sets. Consider therefore the following definition.

Definition 5.3 Let (X, \leq) be a poset. A relation $R \subseteq X^2$ is called an increasing relation if for all $x, y, z \in X$, xRy and $y \leq z$ implies xRz . That is, R is increasing if the set $\{y \mid xRy\}$ is an up-set. Dually, $Q \subseteq X^2$ is a decreasing relation if for all $x, y, z \in X$, xQy and $z \leq y$ implies xQz .

We refer to the above as monotonicity conditions for R and Q . We put off the question of how the relations must interact with the topology for the

moment, and consider the following: Now that we know that we need join- and meet-hemimorphisms on the lattice side, and that we need increasing- and decreasing relations on the topological side, how do we go about obtaining one from the other?

Consider the ordered relational structure (X, \leq, R, Q) , where $R \subseteq X^2$ is increasing and $Q \subseteq X^2$ is decreasing. Denote by $\mathcal{O}(X)$ the set of all up-sets of X . It is then readily seen that $\mathcal{O}(X)$ is a bounded distributive lattice with the meet and join given by intersection and union respectively. In this lattice the top and bottom elements are given by X itself and the empty set respectively. We are going to show that R and Q determine hemimorphisms on $\mathcal{O}(X)$. For any $Y \in \mathcal{O}(X)$, define

- $\diamond_R(Y) := \{y \in X \mid \exists x \in X, xRy \text{ and } x \in Y\}$
- $\square_Q(Y) := \{y \in X \mid \forall x \in X, xQy \Rightarrow x \in Y\}$.

Then $\diamond_R(Y) \in \mathcal{O}(X)$ since $x \in \diamond_R(Y)$ and $x \leq y$ imply $\exists w \in Y$ such that wRx and $x \leq y$. Hence $\exists w \in Y$ such that wRy since R is an increasing relation showing that $y \in \diamond_R(Y)$. Also $\square_Q(Y) \in \mathcal{O}(X)$. Indeed, $x \in \square_Q(Y)$ with $x \leq y$ and zQy implies that $x \in \square_Q(Y)$ and zQx since Q is decreasing. Hence $z \in Y$ as required. Therefore \diamond_R and \square_Q are both in fact operators on $\mathcal{O}(X)$. We stick with Goldblatt's wording and call $\diamond_R(Y)$ the existential image of Y and $\square_Q(Y)$ the universal image, where $Y \in \mathcal{O}$. Before we prove that these images are indeed hemimorphisms we note that the choice to define these mappings on $\mathcal{O}(X)$ is not accidental since the collection of clopen up-sets of X (which is used for the dual lattice) is a subfamily of the collection of up-sets of X . The next lemma shows that these operators are in fact hemimorphisms.

Lemma 5.4 Let (X, \leq, R, Q) be an ordered relational structure, with R (resp. Q) being an increasing (resp. decreasing) binary relation on X . Then \diamond_R is a join-hemimorphism and \square_Q is a meet-hemimorphism.

Proof. We first notice that in the bounded distributive lattice $\mathcal{O}(X)$, the top element is X itself and the bottom element is \emptyset . Hence $\diamond_R(\emptyset) = \{y \in X \mid \exists x \in X \text{ with } xRy \text{ and } x \in \emptyset\} = \emptyset$. Thus \diamond_R preserves the bottom element. Next we want to show that $\diamond_R(U \cup V) = \diamond_R(U) \cup \diamond_R(V)$, since union is the join operation in $\mathcal{O}(X)$. Thus let $w \in \diamond_R(U \cup V)$. Thus there exists an $x \in X$ with xRw and $x \in U \cup V$. Hence $x \in U$ or $x \in V$. If $x \in U$, then $w \in \diamond_R(U)$ and if $x \in V$, then $w \in \diamond_R(V)$. Thus $\diamond_R(U \cup V) \subseteq \diamond_R(U) \cup \diamond_R(V)$. Conversely, if $w \notin \diamond_R(U \cup V)$ then $\forall x \in X$ we have that $(x, w) \notin R$ or $x \notin U \cup V$, i.e. $x \notin U$ and $x \notin V$. Thus, $\forall x \in X, ((x, w) \notin R \text{ or } x \notin U) \text{ and } ((x, w) \notin R \text{ or } x \notin V)$. That is, $x \notin \diamond_R(U) \cup \diamond_R(V)$. Hence $\diamond_R(U \cup V) = \diamond_R(U) \cup \diamond_R(V)$.

Now we need to do the meet-hemimorphism case. $\square_Q(X)$ is trivially equal to X . It thus remains to show that $\square_Q(U \cap V) = \square_Q(U) \cap \square_Q(V)$. Let $w \in \square_Q(U \cap V)$. Hence for all $x \in X$ we have that $xQw \Rightarrow x \in U \cap V$. That is, $x \in U$ and $x \in V$. It follows that $w \in \square_Q(U) \cap \square_Q(V)$. Conversely, suppose that $w \notin \square_Q(U \cap V)$. Hence there is an $x \in X$ with xQw but $x \notin U \cap V$. That is, $x \notin U$ or $x \notin V$. Hence $w \notin \square_Q(U)$ or $w \notin \square_Q(V)$. Thus $w \notin \square_Q(U) \cap \square_Q(V)$. \square

We thus see that the structure $(\mathcal{O}(X), \cup, \cap, \emptyset, X, \diamond_R, \square_Q)$ is a distributive lattice with operators. Of course, we are going to insist that there is a topology on the relational structure that we study. Indeed, we are going to look at compact, totally order-disconnected spaces (Priestley spaces) that also satisfy some extra properties. Recall that for a Priestley space X , the collection of clopen up-sets and their complements forms a subbase for the topology. We call the topology generated by using the clopen up-sets as a base the upper topology on X and the topology generated by using the complements of the clopen up-sets as a base the lower topology. We now define the type of relational topological space that we will be working with.

Definition 5.5 An ordered relational space (X, \leq, Ω, R, Q) will be called a relational Priestley space if

1. (X, \leq, Ω) is a Priestley space.
2. If R is increasing, then for any $y \in X$ we have that $R^{-1}(y)$ is closed in the upper topology on X .
3. If Q is decreasing, then for any $y \in X$ we have that $Q^{-1}(y)$ is closed in the lower topology on X .
4. Existential and universal images of clopen up-sets are clopen up-sets.

In order to gain some insight as to why we might be interested in the sets $R^{-1}(y)$ and $Q^{-1}(y)$ we observe that it can be shown that the hemimorphisms \diamond_R and \square_Q may be defined equivalently as $\diamond_R(Y) = \{y \mid R^{-1}(y) \cap Y \neq \emptyset\}$ and $\square_Q(Y) = \{y \mid Q^{-1}(y) \subseteq Y\}$. For the reader who wishes to look ahead, these conditions are needed in the proof of Theorem 5.9. If (X, \leq, Ω, R, Q) is a relational Priestley space, then we denote by $(L_X, \diamond_R, \square_Q)$ the dual lattice of X . Note that the above structure is a sublattice of the lattice that we saw in the paragraph following Lemma 5.4. This is because condition (4) above ensures that the hemimorphisms \diamond_R and \square_Q are closed. This is so since all the elements of L_X are by definition clopen up-sets of X and condition (4) ensures that clopen up-sets are taken to clopen up-sets. We have thus seen that, given a relational Priestley space, we can create a bounded distributive lattice with operators. We now have to do the converse, that is, create a relational Priestley space given a bounded distributive lattice with operators. Fortunately for us, much of the work is already done. We already know how to create a Priestley space given a bounded distributive lattice - simply take the set of prime filters and topologize in the way demonstrated in Section 3. All we have to do now is find a way to move from a hemimorphism to a relation. Hence let (L, \diamond, \square) be a bounded distributive lattice with operators (as usual we refer to it simply as L) - f being a join-hemimorphism and g a meet-hemimorphism. Define $R_\diamond \subseteq X_L^2$ and $Q_\square \subseteq X_L^2$, where X_L is the usual dual space of a bounded distributive lattice as follows :

$$(G, F) \in R_\diamond \Leftrightarrow \diamond(G) \subseteq F$$

and

$$(G, F) \in Q_\square \Leftrightarrow \square^{-1}(F) \subseteq G.$$

Observe that, in the above, $\diamond(G) \subseteq F$ is equivalent to the statement that for all $a \in L$, $a \in G$ implies $\diamond(a) \in F$. Similarly $\square^{-1}(F) \subseteq G$ is equivalent to the statement that for all $a \in L$, $\square(a) \in F$ implies $a \in G$. We note further that R_\diamond is an increasing relation. Indeed, if $(G, F) \in R_\diamond$ and $F \subseteq H$ then $\diamond(G) \subseteq H$. Likewise, Q_\square is a decreasing relation because if $(G, F) \in Q_\square$ and $H \subseteq F$ then for all $a \in L$, $\square(a) \in H \Rightarrow \square(a) \in F$ and hence $a \in G$. Thus we see that the structure $(X_L, \subseteq, \Omega_L, R_\diamond, Q_\square)$ is a totally order-disconnected compact relational space, which we call the dual space of L as usual. Of course, we would like to show that it is a relational Priestley space. As before, then we would like to show that a relational Priestley space is order-homeomorphic to the dual space of its dual lattice, and then that a distributive lattice with operators is isomorphic to the dual lattice of its dual space. We therefore have to show that the dual space $(X_L, R_\diamond, Q_\square)$ possesses the properties stated in Definition 5.5. This is done with the next two results. We shall save a little time however and ‘skip’ a step in the sense that we shall not prove condition (4) of Definition 5.5 directly. We notice the following: At the end of the day we shall have to show that the mapping defined in Theorem 3.8 respects the operations that we endowed our lattice with. That is, if (L, \diamond, \square) is a bounded distributive lattice with operators, with \diamond a join-hemimorphism and \square a meet-hemimorphism, then we want, for any $a \in L$,

1. $\eta(\diamond(a)) = \diamond_{R_\diamond}(\eta(a))$
2. and $\eta(\square(a)) = \square_{Q_\square}(\eta(a))$,

where η is the mapping defined in Theorem 3.8 ($\eta(a) := X_a$ where $X_a = \{F \in X_L \mid a \in F\}$). We see immediately that if we can prove these equations, then condition (4) of Definition 5.5 will be satisfied, since the dual lattice of the dual space is the lattice of clopen up-sets of the dual space. Now clearly conditions (1) and (2) are equivalent to the following statements:

$$X_{\diamond(a)} = \diamond_{R_\diamond}(X_a)$$

and

$$X_{\square(a)} = \square_{Q_\square}(X_a).$$

Let us refer to the above as statement (1*) and statement (2*) respectively. We thus prove the next theorem.

Theorem 5.6 Let (L, \diamond, \square) be a bounded distributive lattice with operators. Then,

1. For any $a \in L$ and $F \in X_L$, $\diamond(a) \in F$ iff there exists $G \in X_L$ such that $(G, F) \in R_\diamond$ and $a \in G$.
2. For any $a \in L$ and $F \in X_L$, $\square(a) \in F$ iff for all $G \in X_L$, $(G, F) \in Q_\square$ implies $a \in G$.

Proof. We see immediately that by the definition of universal and existential images, these conditions are equivalent to conditions (1*) and (2*) above. First, let $a \in L$ and $F \in X_L$. If $(G, F) \in R_\diamond$ and $a \in G$, then $\diamond(G) \subseteq F$, hence $\diamond(a) \in F$. Conversely suppose that $\diamond(a) \in F$. Let $H := \{z \mid \diamond(z) \notin F\}$. Then H is closed under finite joins, for if z_1 and z_2 are in H , then $\diamond(z_1) \notin F$ and

$\diamond(z_2) \notin F$. Then $\diamond(z_1 \vee z_2) = \diamond(z_1) \vee \diamond(z_2) \notin F$ for if $\diamond(z_1) \vee \diamond(z_2) \in F$ then, by the primeness of F , $\diamond(z_1) \in F$ or $\diamond(z_2) \in F$, a contradiction. Also, $0 \in H$. Indeed, \diamond preserves 0, and $\diamond(0) = 0 \notin F$ since F is proper. Further, H is a down-set. Indeed, let $a \leq b$ and $b \in H$. Then $\diamond(a) \leq \diamond(b)$ and $\diamond(b) \notin F$. However, F is an up-set, thus $\diamond(a) \notin F$ and hence $a \in H$. Thus, H is an ideal. Since $a \notin H$, H is proper and hence by Lemma 1.15 there is a prime ideal J such that $H \subseteq J$ and a prime filter G such that $\uparrow a \subseteq G$ where $J \cap G = \emptyset$. That is, $G \cap H = \emptyset$. Now $y \in G \Rightarrow \diamond(y) \in F$ for if not, then $y \in H$ and hence also $y \in H \cap G$, a contradiction. Thus $(G, F) \in R_\diamond$ and $a \in G$, completing the proof of part 1.

For part 2, if $\square(a) \in F$, then whenever $(G, f) \in Q_\square$ where $Q_\square \in X_L$, it follows that $\square^{-1}(F) \subseteq G$, hence $a \in G$. Conversely suppose that $\square(a) \notin F$. Let $E := \{z \mid \square(z) \in F\}$. Then F is closed under finite meets. Indeed, if z_0 and z_1 are such that $\square(z_0) \in F$ and $\square(z_1) \in F$ then $\square(z_0 \wedge z_1) = \square(z_0) \wedge \square(z_1) \in F$ since F is a filter. Further, $1 \in E$ since $\square(1) = 1 \in F$ since F is an up-set. Additionally, since F is an up-set, so is E by the monotonicity of \square . Hence E is a filter which is disjoint from a . By Lemma 1.15, there is a prime filter G which contains E and is disjoint from a . If we understand that E is exactly $\square^{-1}(F)$, it follows that $(G, F) \in Q_g$, completing the proof of 2. \square

All that remains is to show that the pre-images of points are closed in the appropriate weak topology. We do this next.

Theorem 5.7 Let $F \in X_L$. Then

1. $R_\diamond^{-1}(F)$ is closed in the upper topology on X_L .
2. $Q_\square^{-1}(F)$ is closed in the lower topology on X_L .

Proof. We will show that $R_\diamond^{-1}(F)$ is closed by showing that its complement is open (of course in the upper topology). Thus let $G \notin R_\diamond^{-1}(F)$. Hence, $(G, F) \notin R_\diamond$. Hence there exists $a \in G$, $\diamond(a) \notin F$. Thus $G \in X_a$ which is open in the upper topology. Further, $Z \in X_a \Rightarrow Z \notin R_\diamond^{-1}(F)$ because $a \in Z$ and $\diamond(a) \notin F$. Hence G is contained in an open set disjoint from $R_\diamond^{-1}(F)$, showing that $R_\diamond^{-1}(F)$ is closed.

We follow the same plan for part 2. Thus let $G \notin Q_\square^{-1}(F)$. Thus $(G, F) \notin Q_\square$. Hence there exists $a \in L$ with $\square(a) \in F$ but $a \notin G$. Then $G \in X_L \setminus X_a$ which is open in the lower topology. Further, $X_L \setminus X_a$ is disjoint from $Q_\square^{-1}(F)$. Indeed, $Z \in X_L \setminus X_a \Rightarrow a \notin Z$ and $\square(a) \in F$. Thus $(Z, F) \notin Q_\square$. \square

By combining all the previous results we see that we have proved the following theorem.

Theorem 5.8 Let (L, \diamond, \square) be a bounded distributive lattice with operators, \diamond a join-hemimorphism and \square a meet-hemimorphism. Then the dual space X_L is a relational Priestley space and the mapping

$$\eta : L \rightarrow L_{X_L}$$

defined by

$$\eta(a) = X_a$$

is an isomorphism of L onto the dual lattice L_{X_L} of its dual space X_L which respects the operators. That is, for any $a \in L$,

$$\eta(\diamond(a)) = \diamond_{R_\diamond}(\eta(a))$$

and

$$\eta(\square(a)) = \square_{Q_\square}(\eta(a)).$$

One final result remains in this section. We have to show that a relational Priestley space is homeomorphic to the dual space of its dual lattice, and that it is an isomorphism of ordered relational structures.

Theorem 5.9 Let (X, Ω, \leq, R, Q) be a relational Priestley space, with R increasing and Q decreasing. Let $(L_X, \diamond_R, \square_Q)$ be the dual lattice of X and let $(X_{L_X}, \Omega_{L_X}, \leq, R_{\diamond_R}, Q_{\square_Q})$ be the dual space of L_X . Then

$$\varphi : X \rightarrow X_{L_X}$$

defined by

$$\varphi(x) = \{Y \in L_X \mid x \in Y\}$$

is a homeomorphism which preserves the relational structure, that is, for any $x, y \in X$ it follows that

$$R(x, y) \Leftrightarrow R_{\diamond_R}(\varphi(x), \varphi(y))$$

and

$$Q(x, y) \Leftrightarrow Q_{\square_Q}(\varphi(x), \varphi(y)).$$

Proof. All that remains is for us to show that φ respects the relations R and Q , since the results proved in Chapter 3 guarantee the rest. First, define $F_x := \{Y \in L_X \mid x \in Y\}$. We first prove that φ respects R . Thus let $(x, y) \in R$. Then for all $Y \in L_X$, if $Y \in F_x$ then $x \in Y$ hence $y \in \diamond_R(Y)$, thus $\diamond_R(Y) \in F_y$. Hence $\diamond_R(F_x) \subseteq F_y$. Thus $(F_x, F_y) \in R_{\diamond_R}$.

Conversely, suppose that $(x, y) \notin R$ for some $x, y \in X$. Then obviously $x \notin R^{-1}(y)$, but $R^{-1}(y)$ is closed in the upper topology on X , so there is a basic open neighbourhood N of x that is disjoint from $R^{-1}(y)$. By definition of the upper topology, $N \in L_X$. That is, $N \in F_x$ and $R^{-1}(y) \cap N = \emptyset$, so that $y \notin \diamond_R(N)$. Hence $\diamond_R(F_x) \not\subseteq F_y$ hence $(F_x, F_y) \notin R_{\diamond_R}$.

Similarly let $(x, y) \in Q$. Then for all $Y \in L_X$, if $\square_Q(Y) \in F_y$ then $y \in \square_Q(Y)$ thus $x \in Y$, thus $Y \in F_x$. Hence $(F_x, F_y) \in Q_{\square_Q}$.

Conversely, if $(x, y) \notin Q$, then $x \notin Q^{-1}(y)$, thus there is some basic open neighbourhood N of x in the lower topology on X that is disjoint from $Q^{-1}(y)$. Then by definition of the lower topology, $N = -Y$ for some $Y \in L_X$. Hence $Q^{-1}(y) \subseteq Y$ which implies that $y \in \square_Q(Y)$, that is, $\square_Q(Y) \in F_y$. But since $x \notin Y$, $Y \notin F_x$. Hence $(F_x, F_y) \notin Q_{\square_Q}$. \square

As a final comment we note that the addition of operators and relations onto Priestley duality does not destroy the functoriality that exists in the ordinary case. Indeed, we refer the reader to Theorem 2.3.3 on page 196 of [Gol89]. As we stated in the introduction, we shall not venture into the details.

5.2 Relational Representations for Operators defined on Bounded Lattices

In this section we perform a further generalization: we expand on the results of Goldblatt and attempt to demonstrate that an analogous result holds for bounded lattices (which are not necessarily distributive). There have been many attempts to obtain a duality for arbitrary bounded lattices equipped with certain operators, such as those discovered by Hartonas [Har97] and Dunn [Dun97], Gehrke [Geh06] and Harding [Geh01] and the latest approach (which is still in preprint) by Moshier and Jipsen [Jip09]. None of the above approaches are based on the representation discovered by Urquhart (even though they could be said to be inspired by them), and hence none of them specialize to Goldblatt's results. In other previous attempts to achieve the aforementioned result only the techniques of logic were applied, resulting in a somewhat 'crowded' structure. Radzikowska [Rad04], for instance, has investigated this non-topological approach, and it appears that one requires multiple relations in order for the structure to be preserved adequately in this case. Moreover, when one ignores the topology, one only obtains an embedding. We demonstrate in what follows that only one relation per operator is needed.

The reader will notice that the proofs of the lemmas and theorems that follow are very much in the Goldblatt tradition. The concepts of join- and meet-hemimorphisms are carried over directly from the distributive case explored in the previous section. However, where relational Priestley spaces were needed to complete Goldblatt's results, we will require relational Urquhart spaces (a notion which will be made precise later). As the name suggests, we will take Urquhart spaces from Section 4.1 and endow them with relations satisfying certain properties similar to those that Goldblatt used. The hope is then that an analogous result will remain. We show that this is indeed the case. Finally, we go one step further and define a slightly more exotic operator (the so-called sufficiency operator) and investigate whether or not it is possible to define a suitable relation to ensure the representation is obtained in this case as well.

5.2.1 Meet- and Join-hemimorphisms

We begin by investigating a bounded lattice which is endowed with a meet-hemimorphism. Denote this lattice by L and denote the operator by \square . Just as in the distributive case, we want the top element in L to be preserved by \square as well. We should decide which way we will define a relation on the Urquhart space that relates to \square in such a way that the duality is ensured. We take our hints from the work of Goldblatt, and define our relation in the following way: If X_L is the dual space of L , then for maximal pairs $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, we define

$$P_{\square}(x, y) \Leftrightarrow \square^{-1}(y_1) \subseteq x_1.$$

We use the filter part of the maximal pairs in the above definition since, in the previous section, the prime filters were used. The first thing we notice

about this relation is the fact that it is decreasing in the second argument with respect to \leq_1 on X_L and we demonstrate it thus in the following lemma.

Lemma 5.10 P_\square is a decreasing relation in the second argument with respect to \leq_1 on X_L , where X_L is the Urquhart space based on a bounded lattice L equipped with a meet-hemimorphism \square .

Proof. Let $(x, y) \in P_\square$, that is, $\square^{-1}(y_1) \subseteq x_1$ and let $z \leq_1 y$. But $\square^{-1}(z_1) \subseteq \square^{-1}(y_1) \subseteq x_1$, showing that $(x, z) \in P_\square$. \square

Turning our attention to the join-hemimorphism case, we start with a bounded lattice L endowed with a join-hemimorphism \diamond . Then, again following the previous section, we define a relation R on the dual space X_L of L as follows:

$$R_\diamond(x, y) \Leftrightarrow \diamond^{-1}(y_2) \subseteq x_2.$$

This might seem at odds with the earlier argument that stated that we should use the filter part. Indeed, at first glance one might not realize how this comes from the distributive case at all. But since prime filters and prime ideals are set complements of each other, we see that our way of defining a relation from \diamond does specialize to the distributive case. Our way of constructing the relation simply highlights the symmetry with the meet-hemimorphism case more effectively. Indeed, viewed from a different perspective, one might argue that this symmetry provides a motivation for defining the relation in the way we do in the distributive case. The next lemma shows that this relation has a monotonicity property similar to its meet-hemimorphism brother.

Lemma 5.11 Let L be a bounded lattice with a join-hemimorphism \diamond . Then the relation R_\diamond defined by $R_\diamond(x, y)$ iff $\diamond^{-1}(y_2) \subseteq x_2$ is decreasing with respect to \leq_2 in the second argument for any elements $x, y \in X_L$.

Proof. Let $R_\diamond(x, y)$ with $x, y \in X_L$. Let $z \in X_L$ and suppose that $z \leq_2 y$, that is, $z_2 \subseteq y_2$. Now $R_\diamond(x, y)$ implies $\diamond^{-1}(y_2) \subseteq x_2$. Then $\diamond^{-1}(z_2) \subseteq \diamond^{-1}(y_2)$ shows that $R(x, z)$. \square

The reader might wonder why both of the relations that we see here are decreasing, as opposed to the distributive case, where one was increasing and the other decreasing. This can be understood if we recall that \leq_1 and \leq_2 are dual, in the sense that they are inverses of each other, in the distributive case. From now on, we will omit the mention of the argument to which the monotonicity applies, since it will always be the second argument. We further save space by calling a relation \leq_1 -decreasing instead of saying that it is decreasing ‘with respect to \leq_1 ’, and similarly for \leq_2 . We refer to the above as the monotonicity properties that the relations possess.

Now we have to do the opposite, that is, we have to clarify how we are going to define a meet-hemimorphism (resp. a join-hemimorphism) if we are given a decreasing (resp. an increasing) relation. We keep in mind that, in the distributive case, these relations were not defined on an ordinary Priestley space. Indeed, the space needed to satisfy several additional properties. We refer the reader to the definition of a relational Priestley space on page 48. Since we intend to follow an analogous path, the first order of business is to

find an ‘Urquhart analog’ of a relational Priestley space, so that we have the appropriate structure at hand upon which to define our relations. The idea is basically to examine the properties of a relational Priestley space one by one and somehow come up with ‘non-distributive versions’ of them. Of course the base space will still be an Urquhart space, just as the base space for the distributive case was a Priestley space.

We first try to find analogs for the concepts of ‘upper topology’ and ‘lower topology’. Even a cursory glance at the definition of the topology on an Urquhart space will guide the reader to make the same natural decision as the author of this thesis. We recall that definition below:

We defined a topology on X by taking the family:

$$\{-F_a \mid a \in L\} \cup \{-rF_a \mid a \in L\}$$

as a subbasis. Here F_a is not to be confused with F_x as in Theorem 5.9. Then we went on to show that F_a was l -stable and that rF_a was r -stable for any lattice element a . Bearing this in mind then, we call the topology generated by using the family $\{-F_a\}$ as a basis the *left* topology on X . For us to be able to use the aforementioned family as a basis, it needs to be closed under finite intersection. This works because the lr operator, though not additive in general, is additive on the \leq_1 -increasing sets by Lemma 4.3. Note that for each $a \in L$, the set F_a is \leq_1 -increasing by Lemma 4.1 and its l -stability. Similarly rF_a is \leq_2 -increasing. We use this to show closure under finite intersection. For $a, b \in L$,

$$\begin{aligned} -F_a \cap -F_b &= -(F_a \cup F_b) && \text{De Morgan's laws} \\ &= -(lrF_a \cup lrF_b) && \text{each } F_a \text{ is } l\text{-stable} \\ &= -(lr(F_a \cup F_b)) && lr \text{ is additive on } \leq_1\text{-increasing sets} \\ &= -(l(rF_a \cap rF_b)) && \text{Lemma 4.2 property (2)} \\ &= -(F_a \vee F_b) && \text{definition of join} \end{aligned}$$

Then by invoking Lemma 4.10 we have that $F_a \vee F_b$ is doubly-closed and stable. Finally, by Lemma 4.13, $F_a \vee F_b = F_c$ for some $c \in L$. Thus the family $\{-F_a\}_{a \in L}$ is a basis for a topology on X_L . The following dual argument shows that the family $\{-rF_a\}$ can also be used as a basis for a topology on X_L , called the *right* topology. For $a, b \in L$,

$$\begin{aligned} -rF_a \cap -rF_b &= -(rF_a \cup rF_b) && \text{De Morgan's laws} \\ &= -(rl(rF_a) \cup rl(rF_b)) && rF_a \text{ is } r\text{-stable} \\ &= -rl(rF_a \cup rF_b) && rl \text{ is additive on } \leq_2\text{-increasing sets} \\ &= -r(lrF_a \cap lrF_b) && \text{Lemma 4.2 property (2)} \\ &= -r(F_a \cap F_b) && F_a \text{ is } l\text{-stable} \end{aligned}$$

Then $F_a \cap F_b$ is doubly-closed and stable by Lemma 4.10 and hence by Theorem 4.13 $F_a \cap F_b = F_c$ for some $c \in L$. Lastly we should decide how we are going to define our hemimorphisms based on the relations we shall start with - again keeping in mind condition (4) on a relational Priestley space. For Y a doubly-closed stable set in an Urquhart space X , define

1. $\Box_P(Y) := [P](Y) = \{x \in X \mid \forall y \in X, yPx \Rightarrow y \in Y\}$ and
2. $\Diamond_R(Y) := l[R]r(Y)$.

These are the preliminary candidates for our meet- and join-hemimorphism respectively. We prove their hemimorphism properties explicitly a little later. For the moment, we have gathered enough information to risk to following definition. The results that follow fully vindicate our intuition in this regard.

Definition 5.12 An ordered relational topological space $(X, \leq_1, \leq_2, \Omega, P, R)$, with both P and R binary relations, is called a relational Urquhart space whenever the following conditions hold:

1. $(X, \leq_1, \leq_2, \Omega)$ is an Urquhart space.
2. P is \leq_1 -decreasing, and for all $y \in X$, $P^{-1}(y)$ is closed in the left-topology on X .
3. R is \leq_2 -decreasing, and for all $y \in X$, $R^{-1}(y)$ is closed in the right-topology on X .
4. If $U \subseteq X$ is doubly-closed and stable, then $\Box_P(U)$ and $\Diamond_R(U)$ are both doubly-closed and stable.

Condition (4) of the above definition implies that the mappings \Box_P and \Diamond_R are automatically operators on L_X , the lattice of doubly-closed stable sets of X . It remains to show then that they are in fact meet- and join-hemimorphisms respectively. We do this in the next two lemmas.

Lemma 5.13 Let $(X, \Omega, \leq_1, \leq_2, P, R)$ be a relational Urquhart space. Then the mapping $\Box_P : L_X \longrightarrow L_X$ is a meet-hemimorphism.

Proof. Let $A, B \in L_X$. Then $\Box_P(A \cap B) = \{b \in X \mid \forall a \in X, aPb \Rightarrow a \in A \cap B\} = \{b \in X \mid \forall a \in X, aPb \Rightarrow a \in A\} \cap \{b \in X \mid \forall a \in X, aPb \Rightarrow a \in B\} = \Box_P(A) \cap \Box_P(B)$. Further, it is clear that the top element of L_X is preserved since the top element is X itself. \square

Lemma 5.14 Let $(X, \Omega, \leq_1, \leq_2, P, R)$ be a relational Urquhart space. Then the mapping $\Diamond_R : L_X \longrightarrow L_X$, is a join-hemimorphism.

Proof. Let $A, B \in L_X$. Then we have

$$\begin{aligned}
 \Diamond_R(A \vee B) &= l[R]r(A \vee B) && \text{definition of } \Diamond_R \\
 &= l[R]rl(r(A) \cap r(B)) && \text{definition of join} \\
 &= l[R](r(A) \cap r(B)) && \text{property of } r\text{-stability} \\
 &= l([R]r(A) \cap [R]r(B)) && \text{property of the box operator}
 \end{aligned}$$

Since A and B are l -stable, $r(A)$ and $r(B)$ are r -stable, hence $[R]r(A)$ and $[R]r(B)$ are both r -stable as well. This implies that $[R]r(A) = rl([R]r(A)) = r\Diamond_R(A)$ and that $[R]r(B) = rl([R]r(B)) = r\Diamond_R(B)$.

Thus $\diamond_R(A \vee B) = l(r \diamond_R(A) \cap r \diamond_R(B))$ \square

Now that we have successfully defined the basic structures with which we are going to be working in this section, we can begin to investigate whether or not the dual lattices and dual spaces we are able to construct satisfy our conditions. That is, we would like to see that the dual space of a bounded lattice with operators is indeed a relational Urquhart space. Of course, by Urquhart duality we have condition (1). So it remains to show the rest. Conditions (2) and (3) are shown in the next lemma.

Lemma 5.15 Let (L, \square, \diamond) be a bounded lattice with meet-hemimorphism \square and join-hemimorphism \diamond . Then $P_{\square}^{-1}(y)$ is closed in the left topology on X_L and $R_{\diamond}^{-1}(y)$ is closed in the right topology on X_L , where X_L is the Urquhart space based on L .

Proof. We prove the first part by showing that the complement is open. Thus let $x \notin P_{\square}^{-1}(y)$. That is, $(x, y) \notin P_{\square}$, hence $\square^{-1}(y_1) \not\subseteq x_1$. This implies that there is an $a \in \square^{-1}(y_1)$ but $a \notin x_1$. That is, $\square(a) \in y_1$ but $a \notin x_1$. Thus $x \notin F_a$ or equivalently, $x \in -F_a$. But $-F_a$ is open set in the left topology. It remains to show that this set is disjoint from $P_{\square}^{-1}(y)$. But this is clearly true since if $z \in -F_a$ then $a \notin z_1$. But we still have that $\square(a) \in y_1$. Hence $z \notin P_{\square}^{-1}(y)$, showing that $P_{\square}^{-1}(y)$ is closed in the left topology.

As before, we show that the complement is open. Thus let $x \notin R_{\diamond}^{-1}(y)$. Then, of course, $(x, y) \notin R_{\diamond}$. By definition, this means that $\diamond^{-1}(y_2) \not\subseteq x_2$. Thus there is an $a \in L$ such that $\diamond(a) \in y_2$ but $a \notin x_2$. Hence, $x \notin r(F_a)$ or equivalently, $x \in -r(F_a)$ which is open in the right topology. Further, $R_{\diamond}^{-1}(y) \cap -r(F_a) = \emptyset$ since $z \in -r(F_a) \Rightarrow a \notin z_2$. This together with the fact that $\diamond(a) \in y_2$ shows that $\diamond^{-1}(y_2) \not\subseteq z_2$. Thus $z \notin R_{\diamond}^{-1}(y)$. \square

Thus only condition (4) is left. Instead of proving it directly, we use the following idea: Show that the lattice isomorphism taking a bounded lattice to the dual lattice of its dual space respects the hemimorphisms and condition (4) will be satisfied automatically. This saves a little time, since we were required to prove this preservation anyway. We will elucidate on why the aforementioned reasoning is valid after the following theorems. We prove this preservation theorem in two steps, first for the meet- and then for the join-hemimorphism.

5.2.2 Preservation Results

Theorem 5.16 Let (L, \square) be a bounded lattice with a meet-hemimorphism \square and let L_{X_L} be the dual lattice of its dual space. Then for any $a \in L$,

$$F_{\square(a)} = \square_{P_{\square}}(F_a).$$

Proof. Let $x \in F_{\square(a)}$. Then $\square(a) \in x_1$. Of course we want $x \in \square_{P_{\square}}(F_a)$. But $\square_{P_{\square}}(F_a) = [P_{\square}](F_a)$. Thus let $y P_{\square} x$ for some arbitrary but fixed $y \in X_L$. Then $\square^{-1}(x_1) \subseteq y_1$. But $\square(a) \in x_1 \Rightarrow a \in \square^{-1}(x_1)$, which is contained in y_1 . Thus $a \in y_1$, showing that $x \in \square_{P_{\square}}(F_a)$.

Now for the converse. Suppose $x \notin F_{\square(a)}$. Thus $\square(a) \notin x_1$. Let $E = \{b \mid \square(b) \in x_1\}$. Then E is a filter. Indeed, if b and c are both in E , then $\square(b \wedge c) = \square(b) \wedge \square(c)$ which is in x_1 since x_1 is a filter and is closed under meets. Thus $b \wedge c \in E$. Further, if $b \in E$ and $b \leq c$, then by the monotonicity of \square , $\square(b) \leq \square(c)$. Since x_1 is a filter, and is hence up-closed, $\square(c) \in x_1$. Thus $c \in E$. Now consider the filter-ideal pair $\langle E, \downarrow a \rangle$. By Lemma 4.7 this is contained in some maximal pair $y = \langle y_1, y_2 \rangle$. By definition, $E = \square^{-1}(x_1)$, hence $\square^{-1}(x_1) \subseteq y_1$, showing that $y P_{\square} x$ but $a \notin y_1$. \square

The proof is quite similar for the join-hemimorphism, as the next theorem illustrates.

Theorem 5.17 Let (L, \diamond) be a bounded lattice with a join-hemimorphism \diamond and let L_{X_L} be the dual lattice of its dual space. Then for any $a \in L$,

$$F_{\diamond(a)} = \diamond_{R_{\diamond}}(F_a).$$

Proof. We do this proof in two steps. First we show that $r(F_{\diamond(a)}) = [R_{\diamond}]r(F_a)$. Thus let $x \in r(F_{\diamond(a)})$.

Then $\diamond(a) \in x_2$. We want $x \in [R_{\diamond}]r(F_a) = \{v \mid \forall w, w R_{\diamond} v \Rightarrow a \in w_2\}$. Hence let $y R_{\diamond} x$. Then $\diamond^{-1}(x_2) \subseteq y_2$. But $a \in \diamond^{-1}(x_2)$, thus $a \in y_2$.

Conversely, let $\diamond(a) \notin x_2$. We want $x \notin [R_{\diamond}]r(F_a)$. That is, we want to find a y such that $y R_{\diamond} x$ but $a \notin y_2$. Let $E = \{b \mid \diamond(b) \in x_2\}$. Then E is an ideal. Indeed, if b and c are both in E , then $\diamond(b \vee c) = \diamond(b) \vee \diamond(c)$ which is in x_2 since x_2 is itself an ideal. Thus, $b \vee c \in E$. Also, if $b \in E$ and $c \leq b$ then, by the monotonicity of \diamond , we have $\diamond(c) \leq \diamond(b) \Rightarrow \diamond(c) \in x_2$ since x_2 is itself an ideal. Note also that $a \notin E$ by assumption. Thus consider the filter-ideal pair $\langle \uparrow a, E \rangle$. This pair is contained in a maximal pair $y = \langle y_1, y_2 \rangle$ which satisfies the needed conditions and completes the proof of the first part. For the second part of the proof, we simply apply the function l to both sides of the equation $r(F_{\diamond(a)}) = [R_{\diamond}]r(F_a)$. It then follows, by the fact that F_a is left-stable and by the definition of the diamond, that the theorem is proved. \square

Let us now discuss why the above implies that the dual space of a bounded lattice with operators is indeed a relational Urquhart space. It remains only to check that condition (4) is satisfied. Let us therefore take an element A in the dual lattice L_{X_L} of the dual space of L . By the isomorphism between L and L_{X_L} , there is an element $a \in L$ such that $F_a = A$. We want to ensure that $\square_{P_{\square}}(F_a)$ is indeed an element of L_{X_L} . That is, we want it to be a doubly-closed stable set in X_L . Here we invoke Theorem 5.16, which says that $F_{\square(a)} = \square_{P_{\square}}(F_a)$. But it was assumed that \square is an operator on L , thus $\square(a)$ is some other element of L . Then $F_{\square(a)}$ sends this element to its concrete representative in L_{X_L} , which is precisely a doubly-closed stable set of X_L , completing the argument. The argument is totally symmetrical for the join case. Thus the dual space of a bounded lattice with meet- and join-hemimorphisms is indeed a relational Urquhart space, and Goldblatt's preservation results can be generalized in the way shown. Naturally we also need to show that the relations on the Urquhart spaces are respected by the order-homeomorphism between an

Urquhart space and the dual space of its dual lattice. As with the lattice case, we do this in two parts; one for each type of relation depending whether or not it is \leq_1 -decreasing or \leq_2 -decreasing (or more clearly, one for each relation depending whether or not it relates to the meet- or join-hemimorphism).

Theorem 5.18 Let $(X, \Omega, \leq_1, \leq_2, P, R)$ be a relational Urquhart space. Then for any $x, y \in X$,

$$P(x, y) \Leftrightarrow P_{\square_P}(v(x), v(y))$$

where $v(x) := \langle v_1, v_2 \rangle = \langle \{Y \in L(S) \mid x \in Y\}, \{Y \in L(S) \mid x \in r(Y)\} \rangle$.

Proof. Let $P(x, y)$. Recall that

$$\begin{aligned} P_{\square_P}(v(x), v(y)) &\Leftrightarrow \square_P^{-1}(v_1(y)) \subseteq v_1(x) \\ &\Leftrightarrow \square_P^{-1}\{Y \in L_X \mid y \in Y\} \subseteq \{Y \in L_X \mid x \in Y\}. \end{aligned}$$

Thus let $Y_0 \in \square_P^{-1}\{Y \in L_X \mid y \in Y\}$. Then $y \in \square_P(Y_0) = [P](Y_0)$. Thus every x related to y under P is in Y_0 . Thus $x \in Y_0$ showing that $Y_0 \in \{Y \in L_X \mid x \in Y\}$.

For the converse, suppose not $P(x, y)$. Thus $x \notin P^{-1}(y)$. But since $P^{-1}(y)$ is closed in the left topology on X , its complement is open. Hence there is a basic open set N in the left topology that contains x and is disjoint from $P^{-1}(y)$. By definition of the left topology, $N = -Y_0$ for some doubly-closed stable set Y_0 . We want not $P_{\square_P}(v(x), v(y))$, that is, we want to find a doubly-closed stable set Y such that $y \in \square_P(Y)$ but $x \notin Y$. We show that Y_0 satisfies these conditions. Indeed, since $x \in -Y_0$, we trivially have that $x \notin Y_0$. To show the remaining part, recall that $P^{-1}(y) \cap -Y_0 = \emptyset$, hence $P^{-1}(y) \subseteq Y_0$. Thus every element related to y under P is in Y_0 . Thus $y \in \square_P(Y_0)$, completing the proof. \square

Theorem 5.19 Let $(X, \Omega, \leq_1, \leq_2, P, R)$ be a Relational Urquhart space. Then for any $x, y \in X$,

$$R(x, y) \Leftrightarrow R_{\diamond_R}(v(x), v(y))$$

where $v(x) := \langle v_1, v_2 \rangle = \langle \{Y \in L(S) \mid x \in Y\}, \{Y \in L(S) \mid x \in r(Y)\} \rangle$.

Proof. The proof strategy is a carbon copy of the proof above. Let $R(x, y)$.

$$\begin{aligned} R_{\diamond_R}(v(x), v(y)) &\Leftrightarrow \diamond_R^{-1}(v_2(y)) \subseteq v_2(x) \\ &\Rightarrow \diamond_R^{-1}\{Y \in L_X \mid y \in r(Y)\} \subseteq \{Y \in L_X \mid x \in r(Y)\}. \end{aligned}$$

Thus let $Y_0 \in \diamond_R^{-1}\{Y \in L_X \mid y \in r(Y)\}$. Then $y \in r(\diamond_R(Y_0)) = rl[R]r(Y_0) = [R]rY_0$ since the box operator preserves r -stability. Hence everything related to y under R is in $r(Y_0)$, hence x is in $r(Y_0)$.

Conversely, suppose not $R(x, y)$. That is, $x \notin R^{-1}(y)$. But $R^{-1}(y)$ is closed in the right topology on X . Hence there is a basic open set N in this topology that contains x but which is disjoint from $R^{-1}(y)$. By definition, $N = -r(Y_0)$

for some doubly-closed stable set Y_0 . To complete the proof, we need a doubly-closed stable set $Y \in \diamond_R^{-1}(v_2(y))$ such that $Y \notin v_2(x)$. We show that Y_0 satisfies these conditions. Indeed, $-r(Y_0) \cap R^{-1}(y) = \emptyset \Rightarrow R^{-1}(y) \subseteq r(Y_0)$ showing that $y \in [R]r(Y_0)$. Finally $x \notin r(Y_0)$ since $x \in -r(Y_0)$, showing that $R_{\diamond_R}(v(x), v(y))$ is false. \square

As a final note, the author has not investigated what (if any) effect the addition of operators on the lattice has on the functoriality that exists in the ordinary bounded lattice case. This is also left for later enquiry.

5.2.3 Representation Theorem for the Sufficiency Operator

In this section we explore an operator that, in a very literal sense, combines the join-preserving and meet-preserving operators that we encountered in the previous chapter. It is called a sufficiency operator and instead of taking joins to joins or meets to meets, it takes joins to meets. The approach is quite synonymous with our earlier approach. Indeed, the proofs and ideas require only a slight adjustment in order for them to apply to this case. Roughly the idea is as follows: Given an abstract bounded lattice L endowed with a sufficiency operator (we will make precise what we mean by this), we can endow the dual space X_L with a binary relation (satisfying some analogous properties to the relations we had before) and conversely, given an Urquhart space X endowed with a binary relation satisfying some properties, we can endow the dual lattice L_X with a sufficiency operator. Finally, as before, these structures are ‘preserved’ by the lattice isomorphism between a bounded lattice and the dual lattice of its dual space, and by the order-homeomorphism between an Urquhart space and the dual space of its dual lattice.

We now make precise what we mean by a sufficiency operator. Our definition is given in terms of a bounded lattice (even though the concept can be naturally extended to any lattice).

Definition 5.20 Let $(L, 0, 1)$ be a bounded lattice. An operator $f : L \longrightarrow L$ is called a sufficiency operator if for any $a, b \in L$

1. $f(1) = 0$ and
2. $f(a \vee b) = f(a) \wedge f(b)$.

In the preservation results from the previous chapter we saw that we needed certain relations on the Urquhart space in order to preserve the join- and meet-preserving structure on the lattice. Here we will also require the services of a relation. Like its cousins it too will need to satisfy a few properties regarding its interaction with the two pre-orders that are present on the Urquhart space. The behaviour of the relation that we will call upon here is only marginally more complicated than the relations we had before. Let us now state precisely how we are going to define this ‘dual relation’.

If $(L, 0, 1, f)$ is a bounded lattice equipped with a sufficiency operator f , then we define a binary relation Q on the dual space of L as follows:

$$(x, y) \in Q \Leftrightarrow f^{-1}(y_1) \subseteq x_2$$

for any $x, y \in X_L$.

We alluded to the properties of this relation in the previous paragraph. This motivates the following lemma.

Lemma 5.21 Let X_L be the dual space of some bounded lattice $(L, 0, 1, f)$ where f is a sufficiency operator. Then Q is \leq_2 -increasing in the first argument and \leq_1 -decreasing in the second argument. That is,

1. $Q_f(x, y)$ and $z \geq_2 x$ imply $Q_f(z, y)$ and
2. $Q_f(x, y)$ and $y \geq_1 z$ imply $Q_f(x, z)$

for some $x, y, z \in X_L$, where Q_f is the relation based on f , as defined above.

Proof. Let $Q_f(x, y)$ and suppose that $z \geq_2 x$. That means that $f^{-1}(y_1) \subseteq x_2$ and $x_2 \subseteq z_2$ which implies that $Q_f(z, y)$. For the second part, suppose that $y \geq_1 z$, that is, $z_1 \subseteq y_1$. Hence $f^{-1}(z_1) \subseteq f^{-1}(y_1) \subseteq x_2$ which shows that $Q_f(x, z)$. \square

The reader will recall that, in addition to various monotonicity properties, the relations based on the join- and meet-hemimorphisms also had to interact with the topology on X in some way. There is no exception here. The following lemma demonstrates the relationship.

Lemma 5.22 Let X_L be the dual space of a bounded lattice L equipped with a sufficiency operator f . Then $Q_f^{-1}(y)$ is closed in the right topology on X_L .

Proof. We show that the complement of $Q_f^{-1}(y)$ is open in the right topology. Thus let $x \notin Q_f^{-1}(y)$. That means that $(x, y) \notin Q_f$. Hence $f^{-1}(y_1) \not\subseteq x_2$. Thus there is some element $a \in L$ such that $f(a) \in y_1$ but $a \notin x_2$. That is, $x \notin r(F_a)$ or $x \in -r(F_a)$ which is a basic open set in the right topology. Further, $-r(F_a) \cap Q_f^{-1}(y) = \emptyset$ since $(z, y) \in Q_f \Rightarrow f^{-1}(y_1) \subseteq z_2 \Rightarrow a \in z_2 \Rightarrow z \in r(F_a)$. Thus $Q_f^{-1}(y)$ is closed in the right topology on X_L . \square

Of course, we should do the converse of the above procedure, that is, we should begin with a relation satisfying the above monotonicity properties and then construct a sufficiency operator based on that relation. As before, this relation will not be defined on an ordinary Urquhart space, so it behooves us to find the appropriate conditions to impose on an Urquhart space. The above lemma shows us how to handle the interaction that the relation should have with the topology, but we still need a way to construct our sufficiency operator from the appropriate relation. We consider the following candidate mapping: For Y a doubly-closed stable set in an Urquhart space X and Q a binary relation on X , define

$$f_Q(Y) = \Box_Q r(Y) := \{x \in X \mid \forall y \in Y, yQx \Rightarrow y \in r(Y)\}.$$

Definition 5.23 An ordered relational topological space $(X, \Omega, \leq_1, \leq_2, Q)$, with Q a binary relation on X , is called a sufficiency Urquhart space if

1. $(X, \Omega, \leq_1, \leq_2)$ is an Urquhart space.
2. $Q^{-1}(y)$ is closed in the right topology on X and Q is \leq_2 -increasing in the first argument and \leq_1 -decreasing in the second argument.
3. If $U \subseteq X$ is a doubly-closed stable set, then $f_Q(U)$ is also doubly-closed and stable.

Condition (3) of the above definition implies that f_Q will automatically be an operator on L_X . The question is, will it be a sufficiency operator? The next lemma shows that this is indeed the case.

Lemma 5.24 Let $(X, \Omega, \leq_1, \leq_2, Q)$ be sufficiency Urquhart space where Q is a relation satisfying the properties of Lemma 5.21. Then the operator f_Q is a sufficiency operator on L_X .

Proof. Let $A, B \in L_X$. Then

$$\begin{aligned}
 f_Q(A \vee B) &= \Box_Q r(A \vee B) && \text{definition on } f_Q \\
 &= \Box_Q r l(rA \cap rB) && \text{definition of join} \\
 &= \Box_Q (rA \cap rB) && \text{property of } r\text{-stability} \\
 &= \Box_Q rA \cap \Box_Q rB && \text{property of the box operator} \\
 &= f_Q(A) \wedge f_Q(B)
 \end{aligned}$$

□

All that is left is for us to be sure that the dual space of a bounded lattice endowed with a sufficiency operator is indeed a sufficiency Urquhart space. All conditions apart from (3) are already satisfied. As in the meet- and join-hemimorphism case, we use the preservation itself to show this. Thus the final step is to prove the analogs of the preservation theorems of the previous chapters.

Theorem 5.25 Let (L, f) be a bounded lattice with f a sufficiency operator and let L_{X_L} be the dual lattice of its dual space. Then for each $a \in L$,

$$F_{f(a)} = f_{Q_f}(F_a)$$

where $F_a = \{x \in X_L \mid a \in x_1\}$.

Proof. Let $y \in F_{f(a)}$. That is, $f(a) \in y_1$. We want $y \in \Box_Q r(F_a)$. Thus let $(x, y) \in Q_f$. Then $f^{-1}(y_1) \subseteq x_2$. Hence $a \in x_2$. Thus $x \in r(F_a)$.

Conversely suppose that $y \notin F_{f(a)}$. That is, $f(a) \notin y_1$. We want $y \notin \Box_Q r(F_a)$. Hence we want to find an x such that $(x, y) \in Q_f$ but with $a \notin x_2$. Consider $E := \{b \mid f(b) = y_1\}$. Then E is an ideal. Indeed, if $b, c \in E$ then $f(b \vee c) = f(b) \wedge f(c)$ which are both in y_1 . But y_1 is a filter, showing that $b \vee c \in E$. Further, if $b \in E$ and $c \leq b$, it follows that $f(b) \leq f(c)$ since f is antitone. But this implies that $f(c) \in y_1$ again since y_1 is a filter, showing that $c \in E$. Note that $a \notin E$ by assumption. Then $\langle \uparrow a, E \rangle$ is a filter-ideal pair, which, by Lemma 4.7, is contained in some maximal pair x . Then $E = f^{-1}(y_1) \subseteq x_2$ but $a \notin x_1$, completing the proof.

□

Theorem 5.26 Let $(X, \Omega, \leq_1, \leq_2, Q)$ be a sufficiency Urquhart space and let X_{L_X} be the dual space of its dual lattice. Then for any $x, y \in X$,

$$Q(x, y) \Leftrightarrow Q_{f_Q}(v(x), v(y))$$

where the map v is defined as in Theorem 4.15.

Proof. Let $Q(x, y)$. Now

$$\begin{aligned} Q_{f_Q}(v(x), v(y)) &\Leftrightarrow f_Q^{-1}(v_1(y)) \subseteq v_2(x) \\ &\Leftrightarrow f_Q^{-1}(\{Y \in L_X \mid y \in Y\}) \subseteq \{Y \in L_X \mid x \in r(Y)\}. \end{aligned}$$

Thus let $Y_0 \in f_Q^{-1}(v_1(y))$, so that $y \in f_Q(Y_0) = \square_Q rY_0$. Since xQy and $y \in \square_Q rY_0$, $x \in rY_0$ showing that $y_0 \in v_2(x)$.

Conversely suppose not $Q(x, y)$. Then $x \notin Q^{-1}(y)$ which is closed in the right topology. Hence there is a basic open set N in this topology which contains x and is disjoint from $Q^{-1}(y)$. By the definition of the right topology, $N = -rY$ for some doubly-closed stable set Y . We want $f_Q^{-1}(v_1(y)) \not\subseteq v_2(x)$. We show that Y is the needed element. Now since $-rY \cap Q^{-1}(y) = \emptyset$, it follows that $Q^{-1}(y) \subseteq rY$ which in turn implies that $y \in \square_Q rY = f_Q(Y)$. But since $x \in -rY$, we have $Y \notin v_2(x)$, completing the proof. \square

Appendix A - Topological Notions

It should be obvious that, in order for us to develop a duality theory between classes of lattices and classes of topological spaces, we should have a basic understanding of the structures in question. In this section, we introduce the reader to the required material. We don't provide proofs for the Lemmas, since most of these results are easy to prove, and may be found in any standard topology text; [Kel75] for instance.

Definition A.1 A set X together with a family of subsets Ω is called a topological space if

1. Ω is closed under arbitrary unions.
2. Ω is closed under finite intersections.
3. X and \emptyset are both in Ω .

The family of sets Ω is called a topology on X and the members of Ω are called open sets.

This definition coincides with the properties of open intervals of the real numbers. That is, when we talk about open sets in the real numbers, we mean those sets that are themselves open intervals, or those which are unions of open intervals. It can easily be shown that the real numbers, in this setting, forms a topological space. We call this the usual topology on the reals. We write X instead of $(X; \Omega)$, when the topology Ω is understood. In topology we would like to do away with the concept of distance and study the structural properties of X in that setting. For a topologist, there is no difference between a doughnut and a teacup! A subset G of a topological space X is said to be closed iff its complement $X \setminus G$ is in Ω . Sets which are both open and closed are called clopen. We shall see later that the clopen sets play a crucial role in the construction of dualities. Indeed, the family of clopen sets of a topological space is a bounded distributive lattice. A topological space is called connected if X and \emptyset are the only clopen sets.

Now, just like in a vector space, where a set of basis vectors can be responsible for generating the vector space, so too in topology it is possible for a specific family of subsets of a topological space X to *generate* the topology.

Definition A.2 A family \mathcal{B} of subsets of a topological space $(X; \Omega)$ is called a basis for Ω if every member of Ω is a union of members from \mathcal{B} . Further, a

family of sets $\mathcal{B} := \{B_i\}_{i \in I}$ is a basis for a topology on a set

$$X = \bigcup_{i \in I} B_i$$

iff for every B_1 and B_2 in \mathcal{B} , $B_1 \cap B_2$ is a union of members from \mathcal{B} .

In particular, every member of the basis is open, and they are sometimes called basic open sets. Sometimes a family of sets \mathcal{S} isn't closed under finite intersections, and in order for us to turn \mathcal{S} into a basis, we first have to consider the family of finite intersections of \mathcal{S} .

Definition A.3 A family of subsets \mathcal{S} of a topological space $(X; \Omega)$ is called a subbasis for Ω iff the family of sets of finite intersections of \mathcal{S} is a basis for Ω .

Of course, as with all areas of mathematics, we wish to study structure-preserving maps between topological spaces. These are the continuous functions. The definition of continuity was chosen so that, in the case of the reals, the notion would coincide with the ordinary definition of continuity in analysis.

Definition A.4 If $(X; \Omega)$ and $(Y; \mathcal{T})$ are topological spaces and $f : X \rightarrow Y$ is a function between them, then f is said to be continuous at a point $x \in X$ iff $\forall V \in \mathcal{T}$ with $f(x) \in V$, $\exists U \in \Omega$ such that $f(U) \subseteq V$. Further, f is said to be continuous if it is continuous at every point of its domain.

The definition of continuity is a little clumsy to use, so we use a Lemma which gives a few equivalent conditions for continuity.

Lemma A.5 Let $f : X \rightarrow Y$ be a function between topological spaces $(X; \Omega)$ and $(Y; \mathcal{T})$. Then the following are equivalent.

1. f is continuous.
2. $f^{-1}(U)$ is open in X whenever U is open in Y .
3. $f^{-1}(U)$ is closed in X whenever U is closed in Y .

Although continuous maps preserve a lot of structure, it is still not a strong enough condition. In order to capture all the structure on a topological space, we need to define the following map.

Definition A.6 Let $(X; \Omega)$ and $(Y; \mathcal{T})$ be topological spaces and $f : X \rightarrow Y$ a continuous mapping between them, then f is said to be a homeomorphism iff f is one-to-one and onto, and $f(U)$ is open in Y whenever U is open in X . In that case, X and Y are said to be homeomorphic.

It turns out that general topological spaces do not have enough structure for them to be useful. In order to create a meaningful duality between two different classes of structures, we need to consider a more restricted class of spaces and in so doing capture the properties we shall require to set up the duality. There are many ways of restricting oneself to various subclasses of topological spaces, among them being those spaces that satisfy the separation axioms. The Hausdorff separation condition turns out to be the most useful one in the context of duality.

Definition A.7 Let $(X; \Omega)$ be a topological space. X is said to be a Hausdorff space iff $\forall x, y \in X$ with $x \neq y$, there exist disjoint open sets U and V , such that $x \in U$ and $y \in V$.

Another very important notion we need to consider is the notion of compactness. Before we can define it, we need to define the notion of an open covering.

Definition A.8 Let $(X; \Omega)$ be a topological space and $\mathcal{U} := \{U_i\}_{i \in I}$ a family of subsets of X , each U_i being open. Then \mathcal{U} is called an open covering for X iff

$$\bigcup_{i \in I} U_i = X.$$

Further, \mathcal{U} has a finite subcovering iff there is a finite subfamily of \mathcal{U} whose union is X .

We are now ready to define compactness.

Definition A.9 A topological space $(X; \Omega)$ is said to be compact iff every open covering of X has a finite subcovering.

As we stated before, homeomorphisms are very important in duality theory, and the following Lemma states that there are many homeomorphisms between compact Hausdorff spaces.

Lemma A.10 Let $(X; \Omega)$ be a compact Hausdorff space. and let $f : X \rightarrow Y$ be a continuous map from X to any other topological space (Y, \mathcal{T}) . Then

1. $f(X)$ is a compact subset of Y .
2. If Y is also a Hausdorff space and f is bijective, then f is a homeomorphism.

Another very useful feature of compactness is that closed subsets of compact spaces are themselves compact. There is one more very important result about subspaces that we need before we can close the book on topology. We will use it in the case of Priestley duality to prove that a Priestley space is compact.

Lemma A.11 (Alexander's Subbasis Lemma). Let $(X; \Omega)$ be a topological space and \mathcal{S} a subbasis for Ω . Then X is compact if every open cover of X by members of \mathcal{S} has a finite subcover.

Lemma A.12 Let X be a directed complete poset, that is, a set where any directed subset of X has a supremum. Then we may define a topology on X , called the Scott topology, via $U \subseteq X$ is open if U is up-closed and has non-empty intersection with any directed set D whose supremum is in U .

Definition A.13 Let X be an ordered topological space. Then a point $x \in X$ is called finite if $\uparrow x$ is open in X .

Definition A.14 A subset A of a topological space is called irreducible if $A \subseteq B \cup C$ implies that $A \subseteq B$ or $A \subseteq C$ for closed sets B and C .

Definition A.15 A topological space is called sober if any closed irreducible set is of the form $\downarrow x$, where we take the ordering on X to be the specialization ordering, for some unique $x \in X$.

Sobriety is, from an order-theoretic point of view, a very convenient condition to have on a topological space, since it can be shown that any sober space is T_0 , thus enabling one to speak meaningfully about the specialization order. We may thus make the convention that in a sober space, unless otherwise indicated, we will mean the specialization order whenever we refer to order-theoretic properties. The following is a nice property of sober spaces, which we state without proof.

Lemma A.16 In a sober space, any directed set D has a supremum which is in the closure of D . Moreover, any continuous function between sober spaces preserves directed suprema.

Bibliography

- [Bir37] Birkhoff, G. 1937. Rings of sets. *Duke Math. Journal*, 3:443-454.
- [Dav90] Davey, B.A. & Priestley, H.A. 1990. Introduction to lattices and order. *Cambridge University Press, New York*.
- [Geh01] Gehrke, M. & Harding, J. 2001. Bounded lattice expansions. *Journal of Algebra*, 238:345-371.
- [Geh06] Gehrke, M. 2006. Generalized Kripke frames. *Studia Logica*, 84(2):241-275.
- [Gol89] Goldblatt, R.I. 1989. Varieties of complex algebras. *Ann. Pure Appl. Logic*, 44:173-242.
- [Gol75] Goldblatt, R.I. 1975. The Stone space of an ortholattice. *Bulletin of the London Mathematical Society*, 7:45-48.
- [Hal55] Halmos, P. 1955. Algebraic Logic I. Monadic boolean algebras. *Compositio Math*, 12:217-249.
- [Han83] Hansoul, G. 1983. A duality for boolean algebras with operators. *Algebra Universalis*, 17:34-49.
- [Dun97] Hartonas, C. & Dunn M.J. 1997. Stone duality for lattices. *Algebra Universalis*, 37:391-401.
- [Har97] Hartonas, C. 1997. Duality for lattice-ordered and for normal algebraizable logics. *Studia Logica*, 58:403-450.
- [Har92] Hartung, G. 1992. A topological representation for lattices. *Algebra Universalis*, 17:273-299.
- [Jip09] Jipsen, P. & Moshier, A.M. 2009. Topological duality and lattice expansions part II: Lattice expansions with quasioperators. <http://www1.chapman.edu/~jipsen/preprints/quasioperators2.pdf>
- [JoT51] Jónson, B. & Tarski, A. 1951. Boolean algebras with operators, Part I. *American Journal of Mathematics*, 73:891-939.
- [JoT52] Jónson, B. & Tarski, A. 1952. Boolean algebras with operators, Part II. *American Journal of Mathematics*, 74:127-162.
- [Kel75] Kelley, J.L. 1975. General Topology. *Berlin, New York: Springer Verlag*, 149-151.

- [Mos09] Moshier, A.M. & Jipsen, P. 2009. Topological duality and lattice expansions part I: A topological construction of canonical extensions. <http://www1.chapman.edu/~jipsen/preprints/topologicalDual2.pdf>
- [Nac49] Nachbin, L. 1949. On a characterization of the lattice of all ideals of a boolean ring. *Fundamenta Mathematicae* 36:137-142.
- [Pri70] Priestley, H.A. 1970. Representation of distributive lattices by means of ordered Stone spaces. *Bulletin of the London Mathematical Society*, 2:186-190.
- [Rad04] Düntsch, I. & Orłowska, E. & Radzikowska, A.M. 2004. Relational representation theorems for some lattice-based structures. *Journal of Relational Methods in Computer Science*, 1:132-160.
- [Rew05] Orłowska, E. & Rewitzky, I.M. 2005. Duality via truth: A semantic framework for lattice-based logics. *Logic Journal of the IGPL* 13, 4:467-490.
- [Sto36] Stone, M.H. 1936. The theory of representations of boolean algebras. *Trans. Amer. Math. Soc.*, 40:37-111.
- [Sto37] Stone, M.H. 1937. Topological representation of distributive lattices. *Casposis pro Pestovani Matematiky a Fysiky*, 67:1-25.
- [Tar29] Tarski, A. 1929. Sur les classes closes par rapport à certaines opérations élémentaires. *Fundamenta Mathematicae*, 16:195-197.
- [Urq78] Urquhart, A. 1978. A topological representation theorem for lattices. *Algebra Universalis*, 8:45-58.